Published by Institute of Physics Publishing for SISSA
Received: *September 9, 2005*
Accepted: *December 21, 2005*
Published: *January 17, 2006*

Permutation branes and linear matrix factorisations

Håkon Enger
Department of Physics, University of Oslo
P. O. Box 1048 Blindern, NO-0316 Oslo, Norway
E-mail: hakon.enger@fys.uio.no

Andreas Recknagel
King's College London, Department of Mathematics
Strand, London WC2R 2LS, U.K.
E-mail: anderl@mth.kcl.ac.uk

Daniel Roggenkamp
Department of Mathematical Sciences, University of Durham
South Rd, Durham DH1 3LE, U.K.
E-mail: daniel.roggenkamp@durham.ac.uk

Abstract: All the known rational boundary states for Gepner models can be regarded as permutation branes. On general grounds, one expects that topological branes in Gepner models can be encoded as matrix factorisations of the corresponding Landau-Ginzburg potentials. In this paper we identify the matrix factorisations associated to arbitrary B-type permutation branes.

Keywords: D-branes, Conformal Field Models in String Theory.

Contents

1. Introduction and outline

The study of strings and D-branes on Calabi-Yau spaces is a remarkably rich area. These string compactifications are interesting for phenomenological reasons (in the heterotic string version they come closest to realistic particle spectra; if D-branes are added to the type II version, they lead to $N = 1$ low-energy theories), as well as for mathematical reasons: In the 1980s, string theorists conjectured the existence of mirror symmetry for Calabi-Yau target spaces, which has since been refined, by including D-branes, to what is often called the homological mirror symmetry programme, involving derived categories of coherent sheaves and the Fukaya category on the target manifolds [1].

These conjectures deal with the large-volume regime of string theories on CY target spaces, but they were at least partially inspired by investigations of the stringy regime, where efficient descriptions in terms $N = 2$ world-sheet theories are available, most notably in the form of Gepner models (orbifolds of tensor products of $N = 2$ superminimal models). The connection to sigma-models on Calabi-Yau manifolds can be established via Landau-Ginzburg models [2], the critical points of which have a description in terms of minimal models.

Even though Gepner models [3] are defined in a somewhat abstract way, they are rational conformal field theories, so it is possible to find symmetry-preserving boundary conditions (i.e. D-branes) for them [4]. Using those boundary states and relating open string Witten indices to geometrical intersection forms, the authors of [5] achieved a large-volume interpretation of the CFT boundary conditions; see also [6 − 8] for further studies along these lines. Since minimal models are closely related to $N = 2$ Landau-Ginzburg models, the study of supersymmetric boundary conditions for the latter should from the outset be relevant to the study of D-branes on Calabi-Yau manifolds. Special (linear) boundary conditions for LG models were analysed in [9, 10] and also in [11]. Later, building on early work by Warner [12], a connection between LG boundary conditions and factorisations of the respective LG superpotential into two matrices was established in [13 − 15]. This was motivated by an unpublished proposal due to M. Kontsevich, who suggested that topological B-branes in LG models can be described by matrix factorisations of the respective LG potential, as discussed in great detail in the papers by Orlov [16] and Kapustin and Li [14, 15, 17].

From any $N = 2$ superconformal field theory one can obtain two 2-dimensional topological quantum field theories, the A- and the B-model, by performing the respective twists and restricting the full Hilbert space to the cohomology of the BRST operator, which provides the 'physical states' of the topological model. This is also true for non-conformal $N = 2$ models with unbroken R-symmetries, e.g. for LG models with affine target space and quasi-homogeneous superpotentials (see e.g. [18]), which will be studied in this article.

In B-models of such LG theories on world-sheets with boundary, the matrix factors of the bulk superpotential determine the boundary conditions, and in particular the boundary BRST operators [13], therefore the spectrum of physical open string states.

Significantly, the main pieces of CFT data used by Brunner, Douglas et al. to extract large-volume information in [5] were all taken from the 'topological sector' of the Gepner model: In particular, the intersection form counts open string Ramond ground states (with fermion number), so is precisely given by the Euler number of the cohomology of the boundary BRST operator.

There are some important questions in string theory which are, at present, too hard to answer directly in the CFT, but can be tackled in the simplified framework of topological theories. In particular, one can study deformation away from the "Gepner point", induced by marginal bulk (and boundary) fields. The properties of topological D-branes in deformed backgrounds can be encoded in topological D-brane superpotentials; a mainly perturbative approach was presented in [19], recent progress towards computing exact superpotentials has been made in [20, 21]. These superpotentials provide a very efficient description of D-brane stability, and of characteristic CFT data like the chiral ring structure.

Thus it is of some interest to study which LG boundary conditions (given in the form of matrix factorisations) correspond to the CFT boundary conditions that are known for Gepner models so far. All these are rational boundary states, and can be viewed as permutation branes, as introduced in [22]: Whenever n of the minimal model constituents of the Gepner model have the same level, there is a non-trivial action of the permutation group S_n, which can be used to build boundary states that obey permutation gluing conditions, i.e. where the left-moving super-Virasoro generators of the i^{th} minimal model are glued to the right-moving generators of the $\sigma(i)^{\text{th}}$ model for any permutation $\sigma \in S_n$.

Landau-Ginzburg matrix factorisations that reproduce the topological spectra of D-branes in $N = 2$ minimal models are well-known [14, 15, 17, 23], and from these one can form (orbifolds of) tensor products which reproduce the spectra of the simplest $\sigma = \text{id}$ 'permutation' branes from [4], see e.g. [24] for a detailed discussion. The next-simplest permutation branes involving length two cycles were analysed in [25], where the connection between CFT boundary states and the 'rank one matrix factorisations' discussed in [24] was established by computations of spectra and comparison of various other CFT and LG results. In the present paper, we propose a correspondence between arbitrary permutation branes and a special class of *linear matrix factorisations*, which were studied from a purely algebraic point of view in [26]. A linear matrix factorisation is a decomposition of a homogeneous (degree d) polynomial $W(x_1, \ldots, x_n) = \alpha_0 \cdots \alpha_{d-1}$ into d matrices, each of which is linear in the x_i. Grouping some α_i together, one obtains two-term matrix factorisations $W = p_0 p_1$. The general construction we propose covers the cases discussed before (trivial permutation or cycles of at most length two), but it necessarily involves higher rank matrix factors as soon as the permutation has cycles of length three or more.

The evidence we present in support of the proposed correspondence is partly in the form of computer-algebraic computations for explicit matrix factorisations, leading to topological spectra (in particular to Witten indices) which are compared to results obtained in the corresponding Gepner models; here we make extensive use of the package Macaulay2 [27]. In addition, we present a general derivation of the BRST cohomology for open strings stretching between an arbitrary permutation brane and special $\sigma = \text{id}$ branes (tensor products of minimal model boundary conditions). Employing tools from homological algebra a bit more ingeniously, it should be possible to extend this calculation to arbitrary tensor product branes, but already the special cases considered here should be a sufficient starting point to compute charges for arbitrary permutation branes.

The body of the paper starts with a review of the relation between matrix factorisations and topological LG models; in particular, we spell out how the boundary BRST-cohomology is encoded in Ext-groups. In section 3, we revisit boundary states in minimal models and Gepner models and derive the topological open string spectra of the permutation branes, which in particular yields the Witten index. The main new results are contained in section 4: We first present linear matrix factorisations of Landau-Ginzburg potentials $W = x_1^d + \ldots + x_n^d$ in section 4.1, then formulate a conjecture which of these correspond to the topological permutation branes from the third section. Sections 4.2 and 4.3 contain evidence for this correspondence. Some homological algebra arguments and the Macaulay2 codes together with results on the large-volume Chern characters for permutation branes in the $(k = 3)^5$-Gepner model describing a sigma-model on the quintic threefold in $\mathbb{P}^4$ are collected in the appendix.

Apart from open technical problems like finding simpler and more general proofs for the correspondence between certain linear matrix factorisations and rational Gepner model branes, there are some conceptual, and also some physical questions that would be interesting to study in the future. For example, one could try to exploit the concrete Ext-groups arising in our examples as a starting point of a computation of brane superpotentials. It should also be interesting to compare our description of permutation LG branes to the one given in [28], and to try and extend the present constructions to D-type modular invariants, see [29] for some recent results.

On the whole, it is probably fair to say that the link between topological B-branes in LG models and matrix factorisations on the one hand and boundary CFT on the other is, as yet, rather loose, and that a deeper understanding of the connection would be desirable. For instance, only part of the linear factorisations which are described in section 4.1 actually correspond to permutation branes, and one wonders what CFT boundary conditions the additional factorisations correspond to — if any. They might correspond to non-rational (symmetry-breaking) Gepner boundary states, which so far are not at all under control, and one may hope that matrix factorisations point towards new constructions of CFT boundary conditions. For these and other reasons, it is definitely worth-while to aim at a better understanding of the TFT-CFT interplay.

2. Landau-Ginzburg models and matrix factorisations

In this section, we briefly recall the relation between topological B-type branes in Landau-Ginzburg models and matrix factorisations, and how tools from homological algebra can be used to describe data of topological string theory.

An $N = 2$ supersymmetric Landau-Ginzburg model with target space $\mathbb{C}^n$ on a world-sheet Σ is given by the *bulk action*

$$S_\Sigma = \int_\Sigma d^2x \Big[\partial^\mu \bar{X}^j \partial_\mu X^j - i\, \bar{\psi}^j_- \overrightarrow{\partial_+} \psi^j_- - i\, \bar{\psi}^j_+ \overrightarrow{\partial_-} \psi^j_+ \\ + \frac{1}{4}\, |\partial W|^2 + \frac{1}{2}\, W_{ij} \psi^i_+ \psi^j_- + \frac{1}{2}\, \overline{W}_{ij} \bar{\psi}^i_- \bar{\psi}^j_+ \Big] \tag{2.1}$$

where X^j, $1 \le j \le n$, are bosonic fields, $\psi^j_\pm$ left- and right-moving fermions, $W(X)$ is the Landau-Ginzburg potential, and $W_{ij} := \partial^2 W / \partial X^i \partial X^j$. (The world-sheet carries the 2-d version of the 'mostly minus' metric, i.e. $ds^2 = dt^2 - dx^2$ in local coordinates.)

This action is invariant under the diagonal $N = 2$ supersymmetry transformation as long as the world-sheet has no boundary; for $\partial\Sigma \neq \emptyset$, one adds *boundary terms* [12, 13]

$$S_{\partial\Sigma,\psi} = \frac{i}{4} \sum_j \int_{\partial\Sigma} dx^0 \, \Big[\, \bar{\theta}^j \eta^j - \bar{\eta}^j \theta^j \,\Big] \tag{2.2}$$

(with $\eta := \psi_- + \psi_+$, $\theta := \psi_- - \psi_+$) as well as a term involving additional boundary fermions

π_α, and boundary potentials $p_i^\alpha(X)$

$$S_{\partial\Sigma,\pi} = \sum_{\alpha,j} \int_{\partial\Sigma} dx^0 \Big[i\,\bar\pi^\alpha \partial_0 \pi^\alpha - \frac{1}{2}\,\bar p_0^\alpha p_0^\alpha - \frac{1}{2}\,\bar p_1^\alpha p_1^\alpha + \frac{1}{2}\,\pi^\alpha(\bar\eta^j\,\overline{\partial}_j \bar p_0^\alpha + i\eta^j\,\partial_j p_1^\alpha) - \frac{1}{2}\,\bar\pi^\alpha(\eta^j\,\partial_j p_0^\alpha - i\bar\eta^j\,\overline{\partial}_j \bar p_1^\alpha)\Big] \tag{2.3}$$

In order to preserve diagonal B-type $N = 2$ supersymmetry, the potentials p_i^α (taken to be polynomial in the X^j) have to satisfy the factorisation condition [13]

$$\sum_\alpha p_0^\alpha\, p_1^\alpha = W$$

(up to a possible additive constant on the rhs, which will be set to zero in the following).

These potentials also determine the action of the (boundary contribution to the) *BRST operator*,

$$Q\,X = 0\ , \quad Q\,\pi = p_0\ , \quad Q\,\bar\pi = -ip_1\,.$$

In the topological field theory, physical open string states correspond to cohomology classes of Q.

The space $\mathcal{P}$ on which the boundary fields act is graded by the fermion number, $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1$, and using Clifford algebra anticommutation relations among the boundary fermions

$$\{\,\pi_\alpha,\,\pi_\beta\} = \{\,\bar\pi_\alpha,\,\bar\pi_\beta\} = 0\ , \quad \{\,\pi_\alpha,\,\bar\pi_\beta\} = \delta_{\alpha,\beta}$$

one can view Q as acting on boundary fields

$$\Phi\ =\ \begin{pmatrix} f_{00} & f_{10} \\ f_{01} & f_{11} \end{pmatrix},$$

where $f_{ij} : \mathcal{P}_i \to \mathcal{P}_j$, by graded commutator with the matrix

$$\Theta\ =\ \begin{pmatrix} 0 & p_1 \\ p_0 & 0 \end{pmatrix}$$

(actually, for $\alpha = 1, \ldots, r$, Q is a $2^r \times 2^r$ matrix).

It is straightforward to carry this over to strings stretching between two different branes (where $\Phi : \mathcal{P} \to \widetilde{\mathcal{P}}$ and $Q\,\Phi = \Theta\,\Phi \pm \Phi\,\widetilde\Theta$). Furthermore, one can generalise this view of the BRST cohomology by allowing for matrices p_0, p_1 of arbitrary size, see [30]. In this way, while losing an explicit realisation through a Clifford algebra spanned by LG boundary fermions π_α, one makes contact to general *matrix factorisations*, which are pairs of square matrices $p_i \in \mathrm{Mat}(k, A)$ over the polynomial ring $A = \mathbb{C}[X_j]$ such that

$$p_0\,p_1 = W\,\mathbf{1}_k\,.$$

Note that the physical content of a matrix factorisation is invariant under gauge transformations as formulated e.g. in [31, 32]: Two matrix factorisations (p_0, p_1) and (p_0', p_1') are called *equivalent* if there are two invertible matrices $U, V \in \mathrm{GL}(k, A)$ with

$$U\,p_0\,V^{-1} = p_0' \quad \text{and} \quad V\,p_1\,U^{-1} = p_1'\,. \tag{2.4}$$

Therefore boundary conditions in topological LG models are indeed described by equivalence classes of matrix factorisations. This will be understood implicitly in the following.

A simple way to obtain boundary conditions in certain LG models is by means of the *tensor product construction*. Whenever the superpotential W is a sum of two polynomials in different variables $W(x_1,\ldots,x_n)=W_1(x_1,\ldots,x_m)+W_2(x_{m+1},\ldots,x_n)$ then the LG model with superpotential W is indeed a tensor product of the two LG models with superpotentials W_1, W_2. Therefore it must be possible in this situation to choose boundary conditions in each of these models separately to obtain the "product" boundary condition in the LG model with potential W. It turns out that the corresponding matrix factorisation is the tensor product matrix factorisation: Let (p_0,p_1), (q_0,q_1) be matrix factorisations of W_1 and W_2, respectively, then the tensor product of these is given by the pair of matrices

$$r_0=\begin{pmatrix} p_0\otimes 1 & -1\otimes q_1 \\ 1\otimes q_0 & p_1\otimes 1\end{pmatrix}, \quad r_1=\begin{pmatrix} p_1\otimes 1 & 1\otimes q_1 \\ -1\otimes q_0 & p_0\otimes 1\end{pmatrix}. \tag{2.5}$$

As discussed in [24], (r_0,r_1) indeed gives rise to the open string spaces associated to tensor product boundary conditions.

Invoking some basic notions from homological algebra, we can relate the spaces of topological open string states, i.e. the cohomology of the BRST-operator Q, to certain *Ext-groups*, which will prove useful for calculations later on. To this end, to a matrix factorisation (p_0,p_1) of W of rank k, we associate the A-module $P=\mathrm{coker}(p_1)$ and its A-free resolution

$$0\longrightarrow A^k \xrightarrow{p_1} A^k \longrightarrow P\longrightarrow 0\,. \tag{2.6}$$

Given another matrix factorisation $(\widetilde{p}_0,\widetilde{p}_1)$ of W of rank $\widetilde{k}$, we obtain another module $\widetilde{P}=\mathrm{coker}\,\widetilde{p}_1$ in the same way. It is easy to see that the space of bosonic BRST-cocycles associated to the pair of matrix factorisations (p_0,p_1) and $(\widetilde{p}_0,\widetilde{p}_1)$ is isomorphic to the space of chain maps between the respective resolutions (2.6). (A chain map between two complexes (C_n,∂_n) and $(\widetilde{C}_n,\widetilde{\partial}_n)$ is given by a sequences of maps $f_n: C_n\to\widetilde{C}_n$ satisfying $f_{n-1}\,\partial_n=\widetilde{\partial}_n\,f_n$.) If the two complexes are resolutions of C and $\widetilde{C}$ respectively, one can show that the space of homomorphisms $\mathrm{Hom}(C,\widetilde{C})$ is isomorphic to the space of chain maps between the respective resolutions modulo the space of chain homotopies. (A homotopy between two chain maps f and f' is a sequence of maps $h_n: C_n\to\widetilde{C}_{n+1}$ satisfying $f_n-f'_n=h_{n-1}\,\partial_n+\widetilde{\partial}_{n+1}\,h_n$.) However, one can check that the space of chain homotopies between resolutions (2.6) is only a subspace of the image of the BRST-operator Q (see also below), essentially because these resolutions are "too short". Thus the bosonic part of the BRST-cohomology in general is a quotient of $\mathrm{Hom}_A(P,\widetilde{P})$.

To obtain a better description of the BRST-cohomology, one can use the fact that due to $W\,P=0$, P is also a module over the ring $R=A/(W)$. Since $p_1p_0=W\,\mathrm{id}_{A^k}=p_0p_1$, this module has the 2-periodic R-free resolution

$$\cdots\longrightarrow R^k\xrightarrow{p_1} R^k\xrightarrow{p_0} R^k\xrightarrow{p_1} R^k\longrightarrow P\longrightarrow 0\,. \tag{2.7}$$

This is a complex with $\partial_{2n}=p_0$ and $\partial_{2n-1}=p_1$ for all $n\geq 1$.

Resolutions (whether periodic or not) can be used to calculate the groups $\mathrm{Ext}^i_R(P,\cdot)$. Namely, for two modules M and N over a ring S, $\mathrm{Ext}^i_S(M,N)$ is defined to be the i^{th} right derived functor of the functor $\mathrm{Hom}_S(\cdot,N)$, i.e. given a projective resolution $\cdots \stackrel{\partial_3}{\to} M_2 \stackrel{\partial_2}{\to} M_1 \stackrel{\partial_1}{\to} M_0 \to M \to 0$ of M, it can be calculated as the i^{th} cohomology of the complex

$$0 \longrightarrow \mathrm{Hom}_S(M_0,N) \longrightarrow \mathrm{Hom}_S(M_1,N) \longrightarrow \cdots\,, \tag{2.8}$$

where the maps are induced by the chain maps ∂_i in the resolution of M, namely $f_i \in \mathrm{Hom}_S(M_i,N) \mapsto f_i \circ \partial_{i+1} \in \mathrm{Hom}_S(M_{i+1},N)$.

A perhaps more concrete way to represent Ext-groups is (see e.g. [33]):

$$\mathrm{Ext}^0_S(M,N) = \mathrm{Hom}_S(M,N)\,, \tag{2.9}$$

$$\mathrm{Ext}^i_S(M,N) = \mathrm{coker}\big(\mathrm{Hom}_S(M_{i-1},N) \longrightarrow \mathrm{Hom}_S(K_i,N)\big) \tag{2.10}$$

where $K_i := \mathrm{im}\,\partial_i \subset M_{i-1}$.

We can, however, use (2.8) directly to make contact with the cohomology of the BRST-operator Q associated to matrix factorisations (p_0,p_1) and $(\widetilde{p}_0,\widetilde{p}_1)$ of W: In even degree, $\ker Q$ is isomorphic to the space of maps $f_{00} \in \mathrm{Hom}_R(R^k,R^{\widetilde{k}})$ such that there exists an $f_{11} \in \mathrm{Hom}_R(R^k,R^{\widetilde{k}})$ with $f_{00}p_1 = \widetilde{p}_1 f_{11}$. Likewise, the even degree part of $\mathrm{im}\,Q$ is isomorphic to $\mathrm{Hom}_R(R^k,R^{\widetilde{k}})\circ p_0 + \widetilde{p}_1 \circ \mathrm{Hom}_R(R^k,R^{\widetilde{k}})$. Dividing out the second summand from $\ker Q$ means that we can choose representatives for f_{00}, which are zero on $\mathrm{im}\,\widetilde{p}_1$. This is achieved by passing from $\mathrm{Hom}_R(R^k,R^{\widetilde{k}})$ to $\mathrm{Hom}_R(R^k,\widetilde{P})$ everywhere, and the condition to belong to $\ker(Q)$ becomes $f_{00}\,p_1 = 0$. In these representatives, the remaining part of $\mathrm{im}(Q)$ is just given by $\mathrm{Hom}_R(R^k,\widetilde{P})\circ p_0$ and one easily sees that $\ker(Q)/\,\mathrm{im}(Q)$ can be obtained as the even cohomology of the complex (2.8) with $M=P$ and $N=\widetilde{P}$. Thus, the bosonic part of the BRST-cohomology is isomorphic to $\mathrm{Ext}^{2i}_R(P,\widetilde{P})$ for $i>0$. (Because of the two-periodicity of the resolution (2.7), all these Ext-groups are isomorphic.)

To obtain the odd part of the BRST-cohomology, one can replace (p_0,p_1) by the shifted matrix factorisation $(-p_1,-p_0)$ in the discussion above, and one finds that the odd BRST-cohomology is isomorphic to $\mathrm{Ext}^{2i-1}_R(P,\widetilde{P})$ for $i>0$.

Altogether, we arrive at the statement that the spaces of states of bosonic respectively fermionic open strings in LG models with boundary conditions characterised by matrix factorisations (p_0,p_1), $(\widetilde{p}_0,\widetilde{p}_1)$ of W are given by

$$\mathrm{H}^{\text{even}}(Q) = \mathrm{Ext}^{2i}_R(P,\widetilde{P})\,,\ \mathrm{H}^{\text{odd}}(Q) = \mathrm{Ext}^{2i-1}_R(P,\widetilde{P})$$

for $i>0$, where $P=\mathrm{coker}\,p_1$ and $\widetilde{P}=\mathrm{coker}\,\widetilde{p}_1$ are the R-modules obtained from the respective matrix factorisations. Interchanging p_0 and p_1 amounts to switching to the anti-brane of P and thus exchanging the notions of bosons and fermions in the open string sectors.

This identification of BRST-cohomology with Ext-groups of the modules P, $\widetilde{P}$ allows us to exploit the machinery of homological algebra (in particular long exact sequences in homology induced by short exact sequences of modules) in the analysis of topological open strings in LG models.

JHEP01(2006)087

Let us remark at this point that the modules P, $\widetilde{P}$ are rather special. Eisenbud [34] (see also [35] for a slight generalisation) showed that *all* minimal free resolutions of finitely generated modules over polynomial rings $R = \mathbb{C}[x_1, \ldots, x_n]/(W)$ become 2-periodic after at most n steps. The modules for which a minimal free resolution is 2-periodic from the start — exactly the ones induced by matrix factorisations — are the *maximal Cohen-Macaulay* modules. These have been studied rather extensively in the mathematical literature.

One aspect of LG models which we have not mentioned up to now is that they carry an *action of a discrete group* Γ. Indeed, if the superpotential W is homogeneous of degree d, which we will assume throughout the paper, this group is given by $\Gamma = \mathbb{Z}_d$ and it acts on the bosonic fields by multiplication with a primitive d^{th} root of unity ξ: $X_i \mapsto \xi^t X_i$ for $t \in \mathbb{Z}_d$. This also induces actions on the open string spaces, and the analysis of the respective representations will be useful for the identification of matrix factorisations associated to conformal boundary conditions.

In terms of matrix factorisations, this group action can be formulated as follows [24]: The $\mathbb{Z}_d$-action on the X_i gives the ring R the structure of a $\mathbb{Z}_d$-graded ring (i.e. the ring structure is compatible with the $\mathbb{Z}_d$-action), and one can consider $\mathbb{Z}_d$-graded modules over it. The latter are modules P over R together with representations $\rho : \mathbb{Z}_d \to \mathrm{End}(P)$ of $\mathbb{Z}_d$ on them, which are compatible with the module structure. In particular, the maps p_0 and p_1 of a matrix factorisation $p_1 : P_1 \rightleftarrows P_0 : p_0$ can be taken as maps between $\mathbb{Z}_d$-graded modules (P_0, ρ_0) and (P_1, ρ_1), which also have to be compatible with the grading, i.e. $\rho_1(g) \circ p_0 = p_0 \circ \rho_0(g)$ and $\rho_0(g) \circ p_1 = p_1 \circ \rho_1(g)$ for all $g \in \mathbb{Z}_d$. Pictorially we write this as

$$\overset{\rho_1}{\circlearrowleft} P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \overset{\rho_0}{\circlearrowright} . \tag{2.11}$$

Such graded matrix factorisations then give rise to $\mathbb{Z}_d$-graded Ext-groups, whose gradings specify the corresponding actions on the corresponding open strings states.

Incorporation of the $\mathbb{Z}_d$-action not only provides finer information about the boundary conditions in LG models, namely the respective $\mathbb{Z}_d$-representations on the open string Hilbert spaces, but also allows to carry the treatment of boundary conditions in LG models over to boundary conditions in LG-orbifolds with orbifold group $\mathbb{Z}_d$. The effect of the orbifolding on the LG model is that the respective open string sectors are projected onto $\mathbb{Z}_d$-invariant subspaces. In terms of matrix factorisations this means that the space of open strings in the LG orbifold model is given by the mod-d-degree-0 parts of the Ext-groups describing the corresponding spaces of open strings in the underlying LG model.

LG orbifolds are relevant because of their relation to non-linear sigma models: The $\mathbb{Z}_d$-orbifold of a LG model with homogeneous superpotential W of degree d in n variables corresponds to a non-linear sigma model defined on the hypersurface $X = \{W = 0\} \subset \mathbb{P}^{n-1}$ in projective space [2], as long as X is a Calabi-Yau manifold, which in the situation considered here is the case if $n = d$. The $\mathbb{Z}_d$-action in the LG model appears here as a "remainder" of the $\mathbb{C}^*$-action divided out to obtain the projective hypersurface. In this case one expects that B-type boundary conditions in the LG orbifold also have a geometric interpretation as B-type D-branes in the non-linear sigma model on X. The latter are believed to be described by objects in the bounded derived category of coherent sheaves $\mathcal{D}^b(\mathrm{Coh}(X))$ on X.

3. LG boundary conditions at the conformal point

The critical behaviour of Landau-Ginzburg models with superpotential $W = x_1^{k_1+2} + \ldots + x_n^{k_n+2}$ can be described in terms of tensor products

$$\mathcal{M}_{k_1,\ldots,k_n} = \mathcal{M}_{k_1} \otimes \ldots \otimes \mathcal{M}_{k_n} \tag{3.1}$$

of $N = 2$ minimal models $\mathcal{M}_k$ with A-type modular invariants.

We are interested in B-type boundary conditions for those tensor product theories that preserve the $N = 2$ supersymmetries of each of the minimal model, i.e. satisfy gluing conditions for all of the $N = 2$ super-Virasoro algebras separately. Certainly, there are the obvious ones, namely "tensor products" of boundary conditions of each of the $\mathcal{M}_{k_i}$. However, if some of the factor models are isomorphic, i.e. $k_i = k_j$ for some $i \neq j$, then it is also possible to construct boundary conditions whose gluing conditions permute the $N = 2$ super-Virasoro algebras of the respective models. Such boundary conditions are called *permutation boundary conditions*.

For tensor products of rational CFTs with diagonal modular invariant there is a standard construction for the corresponding permutation boundary states [22]. This construction has to be slightly modified when dealing with B-type gluing automorphisms, with respect to which minimal models are not diagonal. (A somewhat pedestrian approach to tackle similar problems in constructing permutation branes for Gepner models, i.e. orbifolded tensor products of minimal models, was employed in [22].)

The minimal models $\mathcal{M}_k$ are conformal field theories which are rational with respect to the action of an $N = 2$-super Virasoro algebra at central charge $c_k = \frac{3k}{k+2}$. The bosonic part of this super Virasoro algebra can be realised as the coset W-algebra $(\widehat{\mathfrak{su}}(2)_k \oplus \widehat{\mathfrak{u}}(1)_4)/\widehat{\mathfrak{u}}(1)_{2k+4}$. In fact, the respective coset model can be obtained from the $\mathcal{M}_k$ by a non-chiral GSO-projection, see e.g. [36].

The Hilbert space of the $\mathcal{M}_k$ can be decomposed into irreducible highest weight representations of the respective super Virasoro algebra. It is convenient however, to consider the decomposition into irreducible highest weight representations $\mathcal{V}_{[l,m,s]}$ of its bosonic subalgebra

$$\mathcal{H}_k \cong \bigoplus_{[l,m,s]\in\mathcal{I}_k} \mathcal{V}_{[l,m,s]} \otimes \left(\overline{\mathcal{V}}_{[l,m,s]} \oplus \overline{\mathcal{V}}_{[l,m,s+2]}\right) \,, \tag{3.2}$$

where the set of such representations is

$$\mathcal{I}_k = \{(l,m,s)\,|\, 0 \leq l \leq k \,,\, m \in \mathbb{Z}_{2(k+2)} \,,\, s \in \mathbb{Z}_4 \,,\, l+m+s \in 2\mathbb{Z}\}/\sim \tag{3.3}$$

with the field identification $(l,m,s) \sim (k-l, m+k+2, s+2)$.

Apart from the alignment of R- and NS-sectors, the Hilbert space of the tensor product model (3.1) is just given by the tensor product of the individual minimal model Hilbert spaces

$$\mathcal{H}_{k_1,\dots,k_n} = \bigoplus_{\substack{[l_i,m_i,s_i]\in\mathcal{I}_{k_i}\\ s_i-s_j\in 2\mathbb{Z}}} \bigotimes_{i=1}^{n} \mathcal{V}_{[l_i,m_i,s_i]} \otimes \big(\overline{\mathcal{V}}_{[l_i,m_i,s_i]} \oplus \overline{\mathcal{V}}_{[l_i,m_i,s_i+2]}\big) \ . \tag{3.4}$$

The model possesses the symmetry group $\mathbb{Z}_{k+2}\times\mathbb{Z}_2$ whose generators $g\in\mathbb{Z}_{k+2}$ and $h\in\mathbb{Z}_2$ act on $\mathcal{V}_{[l,m,s]}\otimes\overline{\mathcal{V}}_{[l,\overline{m},\overline{s}]}$ by multiplication with $e^{\frac{\pi i}{k+2}(m+\overline{m})}$ and $e^{\frac{\pi i}{2}(s+\overline{s})}$, respectively.

We would like to analyse boundary conditions whose gluing automorphisms permute the $N=2$ algebras of the $\mathcal{M}_{k_i}$ in (3.1), and since only isomorphic $N=2$ algebras can be "glued together", we will restrict ourselves to the case of tensor products of n identical minimal models $\mathcal{M}_k$, i.e. $k_i=k$ for all i.

We first review the construction for the trivial permutation $\sigma=\mathrm{id}$, which is also discussed in great detail in [37]. Already in this case, where the gluing conditions factorise into n independent ones, the corresponding boundary states are not just tensor products of single minimal model boundary states because of the sector alignment. Furthermore, one has to take into account that a single minimal model is not diagonal with respect to the B-type gluing automorphism.

3.1 Trivial permutation

The B-type gluing automorphism τ_B of the $N=2$ superconformal algebra induces an isomorphism $\mathcal{V}_{[l,m,s]}\xrightarrow{\sim}\mathcal{V}_{[l,-m,-s]}$ of the minimal model representations. Therefore a sector

$$\mathcal{V}_{[l,m,s]}\otimes\overline{\mathcal{V}}_{[l,m,\overline{s}]}\subset\mathcal{H}_k \tag{3.5}$$

in a single minimal model gives rise to an Ishibashi state satisfying B-type gluing conditions iff $[l,m,s]=\tau_B[l,m,\overline{s}]=[l,-m,-\overline{s}]$. Thus, in a single minimal model there are Ishibashi states

$$|[l,0,s]\rangle\!\rangle_B \ , \quad \text{for all } [l,0,s]\in\mathcal{I}_k \ . \tag{3.6}$$

Note however, that one can also introduce Ishibashi states $|[l,m,s]\rangle\!\rangle_B$ for $[l,m,s]\in\mathcal{I}_k$ with $m\neq 0 \bmod (k+2)$, if one allows for twists with respect to the $\mathbb{Z}_{k+2}$-symmetry of the model, *cf.* [36]. These additional Ishibashi states appear in the decomposition of twisted boundary states, whose existence can be understood as follows: Since the boundary states built from the Ishibashi states (3.6) are invariant under the group $\mathbb{Z}_{k+2}$, the latter also acts on the respective open string sectors. Thus we can insert a $\mathbb{Z}_{k+2}$-generator in a trace over an open sector, and by means of modular transformation this can be rewritten as an overlap of twisted boundary states[1]; for more details about this point see e.g. [38]. With

[1]Even though these are not states in the Hilbert space of the original model, they can nevertheless be used to describe the corresponding correlation functions.

g being a generator of $\mathbb{Z}_{k+2}$, the g^{-t}-twisted boundary states are given by

$$\begin{aligned}
\|[L,M,S]\rangle\!\rangle_{B,t} &= \frac{1}{\sqrt{k+2}} \sum_{a\in\mathbb{Z}_{k+2}} \sum_{[l,m,s]\in\mathcal{I}_k} e^{\frac{2\pi i}{k+2}a(m-t)} \frac{S_{[L,M,S],[l,m,s]}}{\sqrt{S_{\Omega,[l,m,s]}}} |[l,m,s]\rangle\!\rangle_B \\
&= \frac{1}{\sqrt{k+2}} \sum_{a\in\mathbb{Z}_{k+2}} \sum_{[l,m,s]\in\mathcal{I}_k} e^{-\frac{2\pi i}{k+2}at} \frac{S_{[L,M+2a,S],[l,m,s]}}{\sqrt{S_{\Omega,[l,m,s]}}} |[l,m,s]\rangle\!\rangle_B \\
&= \sqrt{k+2} \sum_{[l,t,s]\in\mathcal{I}_k} \frac{S_{[L,M,S],[l,t,s]}}{\sqrt{S_{\Omega,[l,t,s]}}} |[l,t,s]\rangle\!\rangle_B \,,
\end{aligned} \tag{3.7}$$

where in the last line it is summed over all l, s such that $[l,t,s] \in \mathcal{I}_k$, and

$$\begin{aligned}
S_{[L,M,S],[l,m,s]} &= \frac{e^{-\frac{i\pi}{2}Ss}}{\sqrt{2}} \frac{e^{\frac{i\pi}{k+2}Mm}}{\sqrt{k+2}} S_{L,l} \quad \text{and} \\
S_{L,l} &= \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}(L+1)(l+1)\right)
\end{aligned} \tag{3.8}$$

are the modular S-matrices of the $N=2$ minimal models and the $\widehat{\mathfrak{su}}(2)_k$-WZW models respectively, and where $\Omega = [0,0,0]$ denotes the minimal model vacuum representation. In an untwisted boundary state, all the twisted Ishibashi states are projected out, and the label M determines the representations of $\mathbb{Z}_{k+2}$ on the open string Hilbert spaces. In particular, B-type boundary states in minimal models are labelled by $[L,M,S] \in \mathcal{I}_k$.

The spectra of open string states with corresponding boundary conditions can easily be obtained from the overlaps of the respective boundary states. The bosonic part of open string states with boundary conditions corresponding to $\|[L,M,S]\rangle\!\rangle_B$ and $\|[L',M',S']\rangle\!\rangle_B$ is described by the overlap of $\|[L,M,S]\rangle\!\rangle_B$ and $\|[L',M',S']\rangle\!\rangle_B$, whereas the fermionic part of the spectrum is determined by the overlap of $\|[L,M,S]\rangle\!\rangle_B$ with the boundary state $\|[L',M',S'+2]\rangle\!\rangle_B$, which is obtained from $\|[L',M',S']\rangle\!\rangle_B$ by reversing the sign of its RR-part. Insertion of a power of the generator g of the symmetry group $\mathbb{Z}_{k+2}$ in the trace over the open sector is achieved by considering the overlap of the corresponding twisted boundary states. The calculation of the spectra is straightforward and one obtains

$$\begin{aligned}
\mathrm{tr}_{\mathcal{H}^{\mathrm{bos}}_{[L',M',S'],[L,M,S]}}\left(g^t q^{L_0-\frac{c}{24}}\right) &= {}_{-t,B}\langle\!\langle [L',M',S']\| q^{\frac{1}{2}(L_0+\overline{L}_0)-\frac{c}{24}} \|[L,M,S]\rangle\!\rangle_{B,-t} \\
&= \sum_{\substack{[l,m,s]\in\mathcal{I}_k \\ a\in\mathbb{Z}_{k+2}}} \mathcal{N}_{[L',M',S'][L,M+2a,S]}^{\quad [l,m,s]} \, e^{\frac{2\pi i}{k+2}ta} \, \chi_{[l,m,s]}(q)
\end{aligned} \tag{3.9}$$

where $\chi_{[l,m,s]}$ are the characters and $\mathcal{N}$ the fusion rules of the minimal model.

Because of the sector alignment, one cannot obtain boundary conditions in tensor products of minimal models just by tensoring the boundary conditions (3.7) of single minimal models. Instead one has to project the tensor products of minimal model boundary states onto the contributions coming from Ishibashi states that are twisted with respect to the alignment group $\mathbb{Z}_2^{n-1}$. The result can be written as

$$\begin{aligned}
&\|L_1,\dots,L_n,S_1,\dots,S_n,M=\textstyle\sum_i M_i\rangle\!\rangle^{\mathrm{id}}_{B,t} \qquad\qquad (3.10)\\
&\quad = 2^{\frac{1-n}{2}}\sum_{b_2,\dots,b_n\in\mathbb{Z}_2}\|[L_1,M_1,S_1+2\textstyle\sum_i b_i]\rangle\!\rangle_{B,t}\otimes\displaystyle\bigotimes_{i=2}^{n}\|[L_i,M_i,S_i-2b_i]\rangle\!\rangle_{B,t}\\
&\quad = \frac{(2k+4)^{\frac{n}{2}}}{\sqrt{2}}\sum_{\substack{[l_i,t,s_i]\in\mathcal{I}_k\\ s_i-s_j\in 2\mathbb{Z}}}\prod_{i=1}^{n}\frac{S_{[L_i,M_i,S_i][l_i,t,s_i]}}{\sqrt{S_{\Omega[l_i,t,s_i]}}}\bigotimes_{i=1}^{n}|[l_i,t,s_i]\rangle\!\rangle_B\,,
\end{aligned}$$

where now t refers to the twist by g^{-t} with g the generator of the diagonal $\mathbb{Z}_{k+2}\subset\mathbb{Z}_{k+2}^n$ of the product of minimal model symmetry subgroups; as above, in the last line the sum is understood to be taken over l_i, s_i such that $[l_i,t,s_i]\in\mathcal{I}_k$. Note that the boundary state (3.10) only depends on $M=\sum_i M_i$, which again determines the $\mathbb{Z}_{k+2}$-representations in the open sectors. Namely, taking into account the form of the modular S-matrix (3.8), we see that the boundary states depend on the M-labels only through a phase $e^{\frac{i\pi t}{k+2}(\sum_i M_i)}$ multiplying the t-twisted Ishibashi states. Using (3.9), the spectrum of open strings with boundary conditions corresponding to $\|\alpha\rangle\!\rangle=\|_{L_1,\dots,L_n,S_1,\dots,S_n,M=\sum_i M_i}\rangle\!\rangle^{\mathrm{id}}_B$ and $\|\alpha'\rangle\!\rangle=\|_{L'_1,\dots,L'_n,S'_1,\dots,S'_n,M'=\sum_i M'_i}\rangle\!\rangle^{\mathrm{id}}_B$ can be easily determined

$$\begin{aligned}
&\mathrm{tr}_{\mathcal{H}^{\mathrm{bos}}_{\alpha'\alpha}}\left(g^t q^{L_0-\frac{c}{24}}\right) = {}_{-t}\langle\!\langle\alpha'\|q^{\frac12(L_0+\overline{L}_0)-\frac{c}{24}}\|\alpha\rangle\!\rangle_{-t} \qquad\qquad (3.11)\\
&\quad=\sum_{\substack{a_i\in\mathbb{Z}_{k+2}\\ b_i\in\mathbb{Z}_2}}\sum_{[l_i,m_i,s_i]\in\mathcal{I}_k} e^{\frac{2\pi i}{k+2}t\sum_i a_i}\prod_{i=1}^{n}\chi_{[l_i,m_i,s_i]}(q)\\
&\qquad\qquad\times\mathcal{N}_{[L'_1,M'_1,S'_1][L_1,M_1+2a_1,S_1+2\sum_i b_i]}^{\;[l_1,m_1,s_1]}\prod_{i=2}^{n}\mathcal{N}_{[L'_i,M'_i,S'_i][L_i,M_i+2a_i,S_i-2b_i]}^{\;[l_i,m_i,s_i]}\,.
\end{aligned}$$

The respective fermionic spectrum can be obtained from the bosonic one by shifting an odd number of S'_i by 2. Formula (3.11) gives the expected spectrum for tensor product boundary conditions. Namely, the sector alignment, i.e. the sum over the b_i, ensures that the corresponding space of bosonic (fermionic) open strings is given by the tensor product of all combinations of bosonic and an even (odd) number of fermionic open string spaces of the individual models.

After this short review of product boundary conditions in tensor products of minimal models let us return to permutation boundary conditions.

3.2 Non-trivial permutation

In this section we will present B-type boundary conditions in $\mathcal{M}_k^{\otimes n}$ which preserve all the individual $N=2$ algebras of the minimal models in a different manner. Namely we impose gluing conditions which relate the holomorphic algebra of the i^{th} minimal model to the antiholomorphic one of the $\sigma(i)^{\text{th}}$ minimal model, where $\sigma\in S_n$ is a permutation.

Since every permutation can be written as a product of cyclic permutations, we restrict our discussion to cyclic permutations $\sigma:(1,\dots,n)\mapsto(2,\dots,n,1)$ of the n factor models. The treatment can easily be carried over to the general situation.

To construct boundary states satisfying σ-permuted B-type gluing conditions, we first of all determine the respective Ishibashi states. The t-twisted sector

$$\mathcal{V}_{[l_1,m_1,s_1]} \otimes \ldots \otimes \mathcal{V}_{[l_n,m_n,s_n]} \otimes \overline{\mathcal{V}}_{[l_1,m_1-2t,\bar{s}_1]} \otimes \ldots \otimes \overline{\mathcal{V}}_{[l_n,m_n-2t,\bar{s}_n]} \subset \mathcal{H}^t_{k,\ldots,k} \tag{3.12}$$

gives rise to an Ishibashi state with respect to the σ-permuted B-type gluing automorphism iff

$$[l_i, m_i, s_i] = \tau_B[l_{i+1}, m_{i+1} - 2t, \bar{s}_{i+1}] = [l_{i+1}, -m_{i+1} + 2t, -\bar{s}_{i+1}] \tag{3.13}$$

for all $i \in \mathbb{Z}_n$. We can choose representatives so that this condition becomes

$$l_i = l_1\,, \ m_{2i+1} = m\,, \ m_{2i} = 2t - m\,, \ \ \bar{s}_i = -s_{i+1} \ \text{ for all } \ i\,. \tag{3.14}$$

For n *odd*, we obtain $m_i = t$ for all i, whereas for n even, there are more solutions, namely $m_{2i+1} = m$ for all i and $m_{2i} = 2t - m$. Thus, for *odd* n there are t-twisted Ishibashi states

$$|[l, t, s_1, \ldots, s_n]\rangle\rangle^\sigma \quad \text{for all } \ [l, t, s_1, \ldots, s_n] \in \mathcal{I}_{k,n}\,, \tag{3.15}$$

where $\mathcal{I}_{k,n} = \{(l, m, s_1, \ldots, s_n)\,|\,[l, m, s_i] \in \mathcal{I}_k\}/\sim$ with $(l, m, s_1, \ldots, s_n) \sim (k - l, m + k + 2, s_1 + 2, \ldots, s_n + 2)$. For *even* n on the other hand there exist t-twisted Ishibashi states

$$|[l, m, 2t - m, s_1, \ldots, s_n]\rangle\rangle^\sigma \quad \text{for all } \ [l, m, s_1, \ldots, s_n] \in \mathcal{I}_{k,n}\,. \tag{3.16}$$

Because of this difference between the case of B-type permutation boundary conditions involving permutations of even and odd cycle length, we will treat them separately in the following.

3.2.1 Odd cycle length

Our ansatz for the σ-permuted B-type boundary states is an adaption of the permutation boundary states for diagonal CFTs [22] to our situation. We have to account for the absence of untwisted Ishibashi states with $m \neq 0$, which can be done similarly to the case of a single minimal model discussed above. Furthermore we have to take care of the fact that the minimal models are non-diagonal. The alignment condition is automatically satisfied for σ-permuted gluing conditions. We define t-twisted boundary states as follows

$$\begin{aligned} &\|L, M, S_1, \ldots, S_n\rangle\rangle^\sigma_{B,t} \\ &\quad := \frac{1}{\sqrt{k+2}} \sum_{a \in \mathbb{Z}_{k+2}} \sum_{\substack{[l,m,s_1] \in \mathcal{I}_k \\ s_i - s_1 \in 2\mathbb{Z}}} e^{-\frac{2\pi i}{k+2} a t} \, \frac{S_{[L,M+2a,S_1][l,m,s_1]}}{\left(S_{\Omega[l,m,s_1]}\right)^{\frac{n}{2}}} \, \frac{e^{-\frac{i\pi}{2}\sum_{i>1} S_i s_i}}{2^{\frac{n-1}{2}}} \\ &\qquad\qquad \times |[l, m, s_1, \ldots, s_n]\rangle\rangle^\sigma_B \\ &\quad = \sqrt{k+2} \sum_{\substack{[l,t,s_1] \in \mathcal{I}_k \\ s_i - s_1 \in 2\mathbb{Z}}} \frac{S_{[L,M,S_1][l,t,s_1]}}{\left(S_{\Omega[l,t,s_1]}\right)^{\frac{n}{2}}} \, \frac{e^{-\frac{i\pi}{2}\sum_{i>1} S_i s_i}}{2^{\frac{n-1}{2}}} \, |[l, t, s_1, \ldots, s_n]\rangle\rangle^\sigma_B \end{aligned} \tag{3.17}$$

Using standard facts about modular S-matrices, it is easy to obtain the spectrum of open strings between two such permutation boundary states on both sides. The computation closely parallels the situation of permutation boundary conditions in diagonal CFTs [22], one merely has to take into account the additional Ishibashi states with $s_i \neq s_j$ and the corresponding phases $e^{-\frac{i\pi}{2}\sum_{i>1} S_i s_i}$ in the boundary states. To deal with them, we parametrise the $[l, m, s_1, \ldots, s_n] \in \mathcal{I}_{k,n}$ as $[l, m, s_1, s_1 + b_2, \ldots, s_n + b_n]$, where $[l, m, s_1]$ runs over $\mathcal{I}_k$ and $b_i \in \mathbb{Z}_2$ are arbitrary. The sum over $b_2, \ldots, b_n \in \mathbb{Z}_2$ is independent of s_1, and the result for the open string spectrum between two σ-permutation branes $\|\alpha\rangle\rangle = \|_{L,M,S_1,\ldots,S_n}\rangle\rangle_B^\sigma$ and $\|\alpha'\rangle\rangle = \|_{L',M',S'_1,\ldots,S'_n}\rangle\rangle_B^\sigma$ is

$$\begin{aligned}\mathrm{tr}_{\mathcal{H}^{\mathrm{bos}}_{\alpha'\alpha}}\left(g^t q^{L_0-\frac{c}{24}}\right) &= {}_{-t}\langle\langle\alpha'\| q^{\frac{1}{2}(L_0+\overline{L}_0)-\frac{c}{24}}\|\alpha\rangle\rangle_{-t} \\ &= \sum_{a\in\mathbb{Z}_{k+2}} \sum_{[l_i,m_i,s_i]\in\mathcal{I}_k} e^{\frac{2\pi i}{k+2}at} \prod_{i=1}^n \chi_{[l_i,m_i,s_i]}(q) \\ &\qquad \times \prod_{i=2}^n \delta^{(2)}_{s_i,S'_i-S_i} \mathcal{N}_{[L',M',\sum_i S'_i][L,M+2a,\sum_i S_i]}^{[l_1,m_1,s_1]*\ldots*[l_n,m_n,s_n]} ,\end{aligned} \tag{3.18}$$

where $*$ denotes fusion of minimal model representations; the fusion rules $\mathcal{N}$ are extended linearly to sums of representations. Shifting an odd number of S-labels of one of the boundary states by 2 yields the fermionic spectrum.

Apart from the open sectors with σ-permuted B-type boundary conditions on both sides, we are also interested in the ones with σ-permuted boundary conditions on one and non-permuted boundary conditions on the other side. For this, we first of all need the overlaps between the corresponding t-twisted Ishibashi states:

$$\begin{aligned}{}_B\langle\langle[l_1', t', s_1']| \otimes \ldots \otimes {}_B\langle\langle[l_n', t', s_n']| q^{\frac{1}{2}(L_0+\overline{L}_0)-\frac{c}{24}} |[l, t, s_1, \ldots, s_n]\rangle\rangle_B^\sigma \\ = \prod_{i=1}^n \left(\delta_{l_i',l}\, \delta_{s_i',s}\, \delta_{s_i,s}\right) \mathrm{tr}_{\mathcal{V}^{\otimes n}_{l,t,s}}\left(\sigma q^{L_0-\frac{c}{24}}\right) ,\end{aligned} \tag{3.19}$$

where σ acts on the tensor product space by permuting the factors. As was noted in [25] for the case $n = 2$, this trace equals the character $\chi_{[l,t,s]}(q^n)$ only up to a phase due to the (not necessarily bosonic) statistics of the respective states. More precisely

$$\chi_{[l,t,s]}(q^n) = \mathrm{tr}_{\mathcal{V}^{\otimes n}_{l,t,s}}\left((-1)^{(1-n)F}\sigma q^{L_0-\frac{c}{24}}\right) = e^{(n-1)\left(\frac{\pi i t}{k+2}-\frac{\pi i s}{2}\right)}\mathrm{tr}_{\mathcal{V}^{\otimes n}_{l,t,s}}\left(\sigma q^{L_0-\frac{c}{24}}\right) . \tag{3.20}$$

Here, $(-1)^F = e^{2\pi i J_0}$, where J_0 denotes the zero mode of the U(1)-current from the $N = 2$ algebra. Using this and the modular transformation properties of the minimal model characters

$$\chi_{[l,t,s]}((Sq)^n) = \sum_{[l',t',s']\in\mathcal{I}_k} S_{[l,t,s][l',t',s']}\, \chi_{[l',t',s']}\left(q^{\frac{1}{n}}\right) , \tag{3.21}$$

the open string spectrum between a σ-permuted boundary state $\|\alpha\rangle\rangle = \|_{L,M,S_1,\ldots,S_n}\rangle\rangle_B^\sigma$ on one side and a non-permuted tensor product boundary state

$\|\alpha'\rangle\!\rangle = \|_{L'_1,\dots,L'_n,S'_1,\dots,S'_n,M'=\sum_i M'_i}\rangle\!\rangle^{\mathrm{id}}_B$ on the other, follows as

$$\begin{aligned}\mathrm{tr}_{\mathcal{H}^{\mathrm{bos}}_{\alpha'\alpha}}\left(g^t q^{L_0-\frac{c}{24}}\right) &= {}_{-t}\langle\!\langle \alpha'\| q^{\frac{1}{2}(L_0+\overline{L}_0)-\frac{c}{24}}\|\alpha\rangle\!\rangle_{-t} \\ &= (k+2)^{\frac{n-1}{2}} \sum_{[l,m,s]\in\mathcal{I}_k} \sum_{a\in\mathbb{Z}_{k+2}} e^{\frac{2\pi i}{k+2}at}\chi_{[l,m,s]}\left(q^{\frac{1}{n}}\right) \\ &\quad \times \mathcal{N}_{[L'_1,M'_1,S'_1]*\dots*[L'_n,M'_1,S'_1][L,M+2a+(1-n),\sum_i S_i+(1-n)]}^{[l,m,s]}\,. \end{aligned} \tag{3.22}$$

Note that the effect of the relative phase between twisted characters and σ-twisted traces in (3.20) on the open spectra (3.22) is a shift of coset-W-algebra representations $[l,m,s] \mapsto [l,m+(1-n),s+(1-n)]$ in the open channel.

3.2.2 Even cycle length

For even cycle length n, we define the t-twisted boundary states

$$\begin{aligned}\|L,M,T,S_1,\dots,S_n\rangle\!\rangle^{\sigma}_{B,t} := &\sum_{[l,m,s_1,\dots,s_n]\in\mathcal{I}_{k,n}} e^{\frac{2\pi i}{k+2}Tt}\,\frac{S_{[L,M-2T,S_1],[l,m,s_1]}}{\left(S_{\Omega,[l,m,s_1]}\right)^{\frac{n}{2}}} \\ &\times \frac{e^{-\frac{i\pi}{2}\sum_{i=2}^n S_i s_i}}{\sqrt{2}^{n-1}}\,|[l,m,2t-m,s_1,\dots,s_n]\rangle\!\rangle^{\sigma}_B\ ; \end{aligned} \tag{3.23}$$

see also [25] for the special case $n=2$. It is now straightforward to calculate the open string spectrum between two such permutation boundary states. For $\|\alpha\rangle\!\rangle = \|_{L,M,T,S_1,\dots,S_n}\rangle\!\rangle^{\sigma}_B$ and $\|\alpha'\rangle\!\rangle = \|_{L',M',T',S'_1,\dots,S'_n}\rangle\!\rangle^{\sigma}_B$ the result is

$$\begin{aligned}\mathrm{tr}_{\mathcal{H}^{\mathrm{bos}}_{\alpha'\alpha}}\left(g^t q^{L_0-\frac{c}{24}}\right) &= {}_{-t}\langle\!\langle \alpha'\| q^{\frac{1}{2}(L_0+\overline{L}_0)-\frac{c}{24}}\|\alpha\rangle\!\rangle_{-t} \\ &= \sum_{[l_i,m_i,s_i]\in\mathcal{I}_k} e^{-\frac{2\pi i}{k+2}t(T-T'+\sum_{i\,\mathrm{even}} m_i)} \prod_{i=1}^n \chi_{[l_i,(-1)^{i+1}m_i,s_i]}(q) \\ &\quad \times \prod_{i=2}^n \delta^{(2)}_{s_i,S'_i-S_i}\mathcal{N}_{[L',M'-2T',\sum_i S'_i][L,M-2T,\sum_i S_i]}^{[l_1,m_1,s_1]*\dots*[l_n,m_n,s_n]}\,. \end{aligned} \tag{3.24}$$

As in the case of odd n, the shift by 2 of an odd number of S-labels in one of the boundary states produces the corresponding fermionic spectrum.

The open string spectrum for σ-permutation boundary conditions $\|\alpha\rangle\!\rangle = \|_{L,M,T,S_1,\dots,S_n}\rangle\!\rangle^{\sigma}_B$ at one end and and tensor product boundary conditions $\|\alpha'\rangle\!\rangle = \|_{L'_1,\dots,L'_n,S'_1,\dots,S'_n,M'=\sum_i M'_i}\rangle\!\rangle^{\mathrm{id}}_B$ at the other can be calculated to be

$$\begin{aligned}\mathrm{tr}_{\mathcal{H}^{\mathrm{bos}}_{\alpha'\alpha}}\left(g^t q^{L_0-\frac{c}{24}}\right) &= {}_{-t}\langle\!\langle \alpha'\| q^{\frac{1}{2}(L_0+\overline{L}_0)-\frac{c}{24}}\|\alpha\rangle\!\rangle_{-t} \\ &= (k+2)^{\frac{n-2}{2}} \sum_{[l,m,s]\in\mathcal{I}_k} \sum_{a\in\mathbb{Z}_{k+2}} e^{\frac{2\pi i}{k+2}t(a-T)}\chi_{[l,m,s]}\left(q^{\frac{1}{n}}\right) \\ &\quad \times \mathcal{N}_{[L'_1,M'_1,S'_1]*\dots*[L'_n,M'_1,S'_1][L,M-2T+2a+(1-n),\sum_i S_i+(1-n)]}^{[l,m,s]}\,. \end{aligned} \tag{3.25}$$

As for the case of odd n, the phase in (3.20) affects the open spectra by a shift in representations $[l,m,s] \mapsto [l,m+(1-n),s+(1-n)]$.

3.3 Topological spectra

The topological open spectra associated to the permutation boundary conditions described in section 3.2 can be read off from the full CFT spectra, by extracting the chiral primary contributions. In a single minimal model $\mathcal{M}_k$, chiral primary fields are given by the highest weight vectors of the representations $[l, l, 0] \in \mathcal{I}_k$, in tensor products of minimal models by tensor products of those. Thus, in the situation when the open sectors carry a representation of the sum of all the $N = 2$ algebras of the individual minimal models, the topological spectra can easily be extracted. Otherwise one has much less control of the representation theory and the identification of chiral primaries can be quite difficult. As far as permutation boundary conditions are concerned, this more complicated situation only occurs in sectors of open strings with different permutation gluing conditions, *cf.* eqs. (3.22), (3.25). These cases will be treated at the end of this section and we start with the cases where the gluing conditions on both sides are twisted by the same permutation.

From now on, we will restrict our considerations to boundary states where all S-labels are even. (S odd merely corresponds to the opposite choice of spin structure.) For a single minimal model, the boundary spectra (3.9) then simplify to

$$\operatorname{tr}_{\mathcal{H}^{\mathrm{bos}}_{[L',M',S'],[L,M,S]}}\left(g^t q^{L_0-\frac{c}{24}}\right) = \sum_{[l,m,0]\in\mathcal{I}_k} e^{\frac{\pi i t}{k+2}(m-M+M')}\chi_{[l,m,0]}(q) \times \left(N_{L'L}^{\ \ l}\ \delta^{(4)}_{S-S',0} + (-1)^t N_{L'L}^{\ \ k-l}\ \delta^{(4)}_{S-S',2}\right), \tag{3.26}$$

where

$$N_{L'L}^{\ \ j} = \begin{cases} 1 \text{ if } |L-L'| \le j \le \min(L+L', 2k-L-L') \text{ and } L+L'+j \in 2\mathbb{Z} \\ 0 \text{ otherwise} \end{cases}$$

denotes the $\widehat{\mathfrak{su}}(2)_k$-fusion rules.

From (3.26) the topological open spectra can be easily read off. There are bosonic topological open strings with boundary conditions corresponding to $\|[L, M, 0]\rangle\rangle_B$ and $\|[L', M', 0]\rangle\rangle_B$ for every $l \in \{0, \ldots, k\}$ such that $N_{L'L}^{\ \ l} = 1$, i.e. for all $l \in \{|L-L'|, |L-L'|+2, \ldots, \min(L+L', 2k-L-L')\}$. Their $\mathbb{Z}_{k+2}$-charges are given by $\frac{1}{2}(l-M+M')$. Likewise, there are fermionic topological open strings with these boundary conditions for every $l \in \{0, \ldots, k\}$ such that $N_{L'L}^{\ \ k-l} = 1$, i.e. for all $l \in k - \{|L-L'|, |L-L'|+2, \ldots, \min(L+L', 2k-L-L')\}$. Their $\mathbb{Z}_{k+2}$-charges are given by $\frac{1}{2}(l-M+M'+k+2)$. (As expected, a shift by 2 in the M or M' shifts the $\mathbb{Z}_{k+2}$-charges by 1.) Thus, the bosonic and fermionic topological Hilbert spaces are given by

$$\mathcal{H}^0\left(\|[L',M',0]\rangle\rangle_B, \|[L,M,0]\rangle\rangle_B\right) \cong \bigoplus_{\substack{l=|L-L'| \\ l+L+L'\in 2\mathbb{Z}}}^{\min(L+L',2k-L-L')} \mathbb{C}_{\frac{1}{2}(l-M+M')}, \tag{3.27}$$

$$\mathcal{H}^1\left(\|[L',M',0]\rangle\rangle_B, \|[L,M,0]\rangle\rangle_B\right) \cong \bigoplus_{\substack{l=|L-L'| \\ l+L+L'\in 2\mathbb{Z}}}^{\min(L+L',2k-L-L')} \mathbb{C}_{\frac{1}{2}(-l-M+M'-2)}, \tag{3.28}$$

where the subscript m of $\mathbb{C}_m$ denotes the respective $\mathbb{Z}_{k+2}$-representation. In particular

$$\begin{aligned}\dim \mathcal{H}^0\left(\|{\scriptstyle [L',M',0]}\rangle\!\rangle_B, \|{\scriptstyle [L,M,0]}\rangle\!\rangle_B\right) &= \dim \mathcal{H}^1\left(\|{\scriptstyle [L',M',0]}\rangle\!\rangle_B, \|{\scriptstyle [L,M,0]}\rangle\!\rangle_B\right) \\ &= \min(L, L', k-L, k-L') + 1 \;.\end{aligned}$$

All this information can be summarised in the bosonic and fermionic topological partition functions of a single minimal model

$$\mathrm{tr}_{\mathcal{H}^0(\|[L',M',0]\rangle\!\rangle_B, \|[L,M,0]\rangle\!\rangle_B)}(g^t) = \sum_{l\in\{0,\dots,k\}} N_{L'L}^{\;\;l} \;\; e^{\frac{2\pi i t}{k+2}\frac{1}{2}(l-M+M')} \,, \tag{3.29}$$

$$\mathrm{tr}_{\mathcal{H}^1(\|[L',M',0]\rangle\!\rangle_B, \|[L,M,0]\rangle\!\rangle_B)}(g^t) = \sum_{l\in\{0,\dots,k\}} N_{L'L}^{\;\;k-l} \;\; e^{\frac{2\pi i t}{k+2}\frac{1}{2}(l-M+M'+k+2)} \,. \tag{3.30}$$

For *tensor product* boundary states $\|\alpha\rangle\!\rangle = \|{\scriptstyle L_1,\dots,L_n, S_1=0,\dots,S_n=0, M=\sum_i M_i}\rangle\!\rangle_B^{\mathrm{id}}$ and $\|\alpha'\rangle\!\rangle = \|{\scriptstyle L'_1,\dots,L'_n, S'_1=0,\dots,S'_n=0, M'=\sum_i M'_i}\rangle\!\rangle_B^{\mathrm{id}}$ we obtain from (3.11)

$$\mathcal{H}^b\left(\|\alpha'\rangle\!\rangle, \|\alpha\rangle\!\rangle\right) = \bigoplus_{\substack{b_1,\dots,b_n\in\mathbb{Z}_2 \\ b+\sum_i b_i\in 2\mathbb{Z}}} \bigotimes_i \mathcal{H}^{b_i}\left(\|{\scriptstyle [L'_i,M'_i,0]}\rangle\!\rangle_B, \|{\scriptstyle [L_i,M_i,0]}\rangle\!\rangle_B\right) \;. \tag{3.31}$$

For *odd cycle length permutation* boundary states $\|\alpha\rangle\!\rangle = \|{\scriptstyle L,M,S_i=0}\rangle\!\rangle_B^{\sigma}$ and $\|\alpha'\rangle\!\rangle = \|{\scriptstyle L',M',S'_i=0}\rangle\!\rangle_B^{\sigma}$ the topological partition functions follow from (3.18)

$$\mathrm{tr}_{\mathcal{H}^0(\|\alpha'\rangle\!\rangle, \|\alpha\rangle\!\rangle)}(g^t) = \sum_{l_i\in\{0,\dots,k\}} N_{L'L}^{\;\;l_1*\dots*l_n} \;\; e^{\frac{2\pi i t}{k+2}\frac{1}{2}\left(\sum_i l_i-M+M'\right)} \,, \tag{3.32}$$

$$\mathrm{tr}_{\mathcal{H}^1(\|\alpha'\rangle\!\rangle, \|\alpha\rangle\!\rangle)}(g^t) = \sum_{l_i\in\{0,\dots,k\}} N_{L'L}^{\;\;(k-l_1)*\dots*l_n} \;\; e^{\frac{2\pi i t}{k+2}\frac{1}{2}\left(\sum_i l_i-M+M'+k+2\right)} \,. \tag{3.33}$$

For *even cycle length permutation* boundary states $\|\alpha\rangle\!\rangle = \|{\scriptstyle L,M,T,S_i=0}\rangle\!\rangle_B^{\sigma}$ and $\|\alpha'\rangle\!\rangle = \|{\scriptstyle L',M',T',S'_i=0}\rangle\!\rangle_B^{\sigma}$ they can be extracted from (3.24) to be

$$\mathrm{tr}_{\mathcal{H}^0(\|\alpha'\rangle\!\rangle, \|\alpha\rangle\!\rangle)}(g^t) = \sum_{l_i\in\{0,\dots,k\}} N_{L'L}^{\;\;l_1*\dots*l_n} \; \delta^{(2k+4)}_{M-M'-2(T-T'),\sum_i(-1)^{i+1}l_i} e^{\frac{\pi i t}{k+2}(\sum_i l_i-M+M')} \,, \tag{3.34}$$

$$\mathrm{tr}_{\mathcal{H}^1(\|\alpha'\rangle\!\rangle, \|\alpha\rangle\!\rangle)}(g^t) = \sum_{l_i\in\{0,\dots,k\}} N_{L'L}^{\;\;(k-l_1)*\dots*l_n} \; \delta^{(2k+4)}_{M-M'-2(T-T'),\sum_i(-1)^{i+1}l_i+k+2} e^{\frac{\pi i t}{k+2}(\sum_i l_i-M+M')} . \tag{3.35}$$

As alluded to above, the extraction of these topological spectra from the corresponding CFT spectra heavily relied on the fact that chiral primary states in tensor products of minimal models are tensor products of minimal model chiral primary states. The Hilbert spaces of open strings satisfying boundary conditions with *different permutations on both sides* however do not carry a representation of the tensor product of the minimal model $N=2$ algebras. Rather, they decompose into twisted representations of a $\mathbb{Z}_n$-orbifold thereof, where $\mathbb{Z}_n$ is generated by the permutation σ.

We can identify the chiral primaries amongst the highest weight vectors of the twisted representations by their characteristic relation between conformal weight h and U(1)-charges q, namely $h = \frac{1}{2}q$ (which holds in unitary theories). Conformal weight $\hat{h}$ and

JHEP01(2006)087

U(1)-charge $\hat{q}$ of the $\mathbb{Z}_n$-twisted representations with character $\chi_{[l,m,s]}(q^{\frac{1}{n}})$ can be expressed in terms of the conformal weight h and U(1)-charge q of the respective representation with character $\chi_{[l,m,s]}(q)$ in the "mother" theory as (for more details on cyclic orbifolds see e.g. [39])

$$\hat{h} = \frac{h}{n} + \frac{c}{24}\left(n - \frac{1}{n}\right), \qquad \hat{q} = q\,. \tag{3.36}$$

The chiral primary condition $\hat{h} = \frac{1}{2}\hat{q}$ can therefore be expressed in terms of h and q as follows

$$h + \frac{1-n}{2}q + \frac{c}{6}\left(\frac{1-n}{2}\right)^2 = \frac{1}{2}\left(q + \frac{c}{3}\left(\frac{1-n}{2}\right)\right). \tag{3.37}$$

This, however, is nothing else than the chiral primary condition for the representation obtained from the original one after spectral flow $\mathcal{U}_\eta$ by $\eta = \frac{1-n}{2}$ units. Namely, conformal weight and U(1)-charge change under the spectral flow $\mathcal{U}_\eta$ as

$$h \mapsto h_\eta = h + \eta q + \frac{c}{6}\eta^2\,, \qquad q \mapsto q_\eta = q + \frac{c}{3}\eta\,. \tag{3.38}$$

The action of this spectral flow on representations is given by

$$\mathcal{U}_{\frac{1-n}{2}}[l,m,s] = [l, m-(1-n), s-(1-n)]\,. \tag{3.39}$$

Therefore, a representation with twisted character $\chi_{[l,m,s]}(q^{\frac{1}{n}})$ is built on a chiral primary highest weight state iff the minimal model representation $[l,m-(1-n),s-(1-n)]$ is built on a chiral primary.

Having managed to identify the chiral primaries in the twisted representations, it is not difficult to extract the topological partition functions between permutation boundary states $\|\alpha\rangle\!\rangle = \|{\scriptstyle L,M,S_i=0}\rangle\!\rangle^\sigma_B$ (for n odd) or $\|\alpha\rangle\!\rangle = \|{\scriptstyle L,M,T,S_i=0}\rangle\!\rangle^\sigma_B$ (for n even) and a tensor production boundary state $\|\alpha'\rangle\!\rangle = \|{\scriptstyle L'_1,\ldots,L'_n,S'_i=0,M'=\sum_i M'_i}\rangle\!\rangle^{\rm id}_B$ from the CFT-spectra[2] (3.22)

$$\mathrm{tr}_{\mathcal{H}^0(\|\alpha'\rangle\!\rangle,\|\alpha\rangle\!\rangle)}(g^t) = (k+2)^{\left[\frac{n-1}{2}\right]} \sum_{l\in\{0,\ldots,k\}} N_{L'_1*\ldots*L'_n L}^{\;l}\; e^{\frac{2\pi it}{k+2}\frac{1}{2}(l-M+M')}\,, \tag{3.40}$$

$$\mathrm{tr}_{\mathcal{H}^1(\|\alpha'\rangle\!\rangle,\|\alpha\rangle\!\rangle)}(g^t) = (k+2)^{\left[\frac{n-1}{2}\right]} \sum_{l\in\{0,\ldots,k\}} N_{L'_1*\ldots*L'_n L}^{\;k-l}\; e^{\frac{2\pi it}{k+2}\frac{1}{2}(l-M+M'+k+2)}\,, \tag{3.41}$$

where $[\,\cdot\,]$ denotes the integer part.

3.4 Boundary states in Gepner models

In this section, we would like to recall briefly how to extract information about Gepner model branes from the boundary states in tensor products of minimal models discussed above.

Gepner models consist of orbifolds of tensor products (3.1) of $N=2$ minimal models coupled to some free external theory, where the orbifold construction implements the GSO-projection of the internal part. The orbifold group is the cyclic group $\Gamma = \mathbb{Z}_H$, generated

[2] Note that the shift (3.39), which is used to identify chiral primaries among the twisted open CFT states, is exactly opposite to the shift in the open spectra produced by the relative phases between twisted characters and σ-twisted traces, *cf.* the end of section 3.2.

by the product of the generators of the $\mathbb{Z}_{k_i+2}$-symmetry groups of the individual minimal models. Hence, $H = \mathrm{lcm}(k_1+2,\ldots,k_n+2)$. If the "Calabi-Yau condition" $\sum_{i=1}^n (k_i+2)^{-1} = 1$ is satisfied, then this model describes a string compactification on the hypersurface in weighted projective space defined by the vanishing of the superpotential $W = x_1^{k_1+2}+\ldots+x_n^{k_n+2}$.

Above, we constructed certain boundary states in tensor products of minimal models, and there is a standard procedure of obtaining boundary conditions in orbifold models from those of the original unorbifolded theories (see the remarks in section 3.1): Starting from a boundary state in the original model which is invariant under the action of the orbifold group Γ, one sums up all the states obtained from it by twisting with elements of Γ, then divides by $\sqrt{|\Gamma|}$ to ensure correct normalisation. Obviously this has the effect of projecting the corresponding open string sectors to the trivial representations of the orbifold group.

In the preceding sections, we obtained such Γ-invariant boundary states in tensor products of minimal models, and we also presented all the twisted boundary states. Summing up all these twisted components, one arrives at the internal parts of the respective boundary states in Gepner models. From the Γ-twisted open partition functions in the tensor products of minimal models determined in section 3.3, one can read off the respective open sectors of the (internal part of the) Gepner model, simply by extracting the Γ-invariant parts.

Boundary states in full Gepner models can be obtained as tensor product of boundary states of the internal and the external theories respectively. However, the alignment of NS- and R-sectors has to be ensured in this construction, intertwining the two factor states in a non-trivial way. Nevertheless, certain "invariants" of boundary conditions, which only depend on the RR- and the NSNS-part of the boundary states separately, factorise into internal and external contributions. This is true in particular for the open string Witten index

$$\begin{aligned} I(\alpha',\alpha) &= {}_{RR}\langle\!\langle \alpha' \| (-1)^{F_L} q^{\frac{1}{2}(L_0+\overline{L}_0)-\frac{c}{24}} \| \alpha \rangle\!\rangle_{RR} \\ &= \dim \mathcal{H}^0\left(\|\alpha'\rangle\!\rangle, \|\alpha\rangle\!\rangle\right) - \dim \mathcal{H}^1\left(\|\alpha'\rangle\!\rangle, \|\alpha\rangle\!\rangle\right) , \end{aligned}$$

where F_L is the holomorphic fermion number on the bulk Hilbert space (see e.g. [5, 11]). Therefore it makes sense to calculate I for the internal part of the Gepner model boundary state alone, i.e. in the orbifold of the tensor products of minimal models.

This index can be calculated easily from the topological open partition functions for tensor product bulk theories. One merely needs to identify the respective Γ-invariant parts of the bosonic and fermionic topological Hilbert spaces and subtract their dimensions.

For example, the Witten index between the tensor product boundary state $\|\alpha'\rangle\!\rangle = \|_{L_i'=0,S_i'=0,M'}\rangle\!\rangle_B^{\mathrm{id}}$ and a permutation brane $\|\alpha\rangle\!\rangle = \|_{L,M,S_i=0}\rangle\!\rangle_B^{\sigma}$ for odd n or $\|\alpha\rangle\!\rangle = \|_{L,M,T,S_i=0}\rangle\!\rangle_B^{\sigma}$ for even n can be obtained by summing over t in eqs. (3.40), (3.41) and dividing by $|\Gamma| = k+2$ (here assuming $k_i = k$ and also $n = k+2$ for notational simplicity). In this way we arrive at

$$I(\alpha',\alpha) = (k+2)^{\left[\frac{n-1}{2}\right]} \left(\delta^{(2k+4)}_{L-M+M',0} - \delta^{(2k+4)}_{L+M-M',2k+2}\right) ,$$

Defining the parameters $\mu(L,M) := \frac{1}{2}(L-M) \in \mathbb{Z}_{k+2}$, and extending them additively to

k	n	L'	L	Δm	$I^{\sigma\sigma}(\alpha',\alpha)$
1	3	0	0		$-3G^2+3G$
2	3	0	0		$-6G^3+6G^2+2G-2$
	3	0	1		$-8G^3+8G$
	3	1	1		$-8G^3+8G^2+8G-8$
	4	0	0	0	$4G^3+10G^2+4G+2$
	4	0	0	1	$-2G^3+4G^2-2G-4$
	4	0	0	2	$-4G^3+2G^2-4G-6$
	4	0	0	3	$-2G^3+4G^2-2G-4$
	4	0	1	0	$8G^2+8G$
	4	0	1	1	$-8G^3-8$
	4	0	1	2	$-8G^3-8$
	4	0	1	3	$8G^2+8G$
	4	1	1	0	$8G^3+16G^2+8G$
	4	1	1	1	$8G^2-8$
	4	1	1	2	$-8G^3-8G-16$
	4	1	1	3	$8G^2-8$
3	3	0	0		$-10G^4+10G^3+5G^2-5$
	3	0	1		$-15G^4+15G^2+5G-5$
	3	1	1		$-15G^4+15G^3+20G^2-20$
	4	0	0	0	$10G^4+20G^3+10G^2+5G+5$
	4	0	0	1	$-5G^4+5G^3-5G^2-10G-10$
	4	0	0	2	$-5G^4+5G^3-5G^2-10G-10$
	4	0	0	3	$10G^3-5G-5$
	5	0	0		$125G^4+125G^3-125G^2-125G$
	5	0	1		$125G^4+250G^3-250G-125$
	5	1	1		$375G^4+250G^3-250G^2-375G$

Table 1: Witten index $I^{\sigma\sigma}$

k	(L'_i)	L	$I^{\mathrm{id}\,\sigma}(\alpha',\alpha)$
1	$(0,0,0)$	0	$-3G^2+3$
2	$(0,0,0)$	0	$-4G^3+4$
	$(0,0,1)$	0	$-4G^3+4G$
	$(0,1,1)$	0	$-4G^3+4G^2+4G-4$
	$(1,1,1)$	0	$8G^2-8$
	$(0,0,0)$	1	$-4G^2+4$
	$(0,0,1)$	1	$-4G^3-4G^2+4G+4$
	$(0,1,1)$	1	$-8G^3+8G$
	$(1,1,1)$	1	$-8G^3+8G^2+8G-8$
	$(0,0,0,0)$	0	$-4G^3+4$
	$(0,0,0,1)$	0	$-4G^3+4G$
	$(0,0,1,1)$	0	$-4G^3+4G^2+4G-4$
	$(0,1,1,1)$	0	$8G^2-8$
	$(1,1,1,1)$	0	$8G^3+8G^2-8G-8$
3	$(0,0,0)$	0	$-5G^4+5$
	$(0,0,1)$	0	$-5G^4+5G$
	$(0,1,1)$	0	$-5G^4+5G^2+5G-5$
	$(1,1,1)$	0	$-5G^4+5G^3+10G^2-10$
	$(0,0,0)$	1	$-5G^3+5$
	$(0,0,1)$	1	$-5G^4-5G^3+5G+5$
	$(0,1,1)$	1	$-10G^4-5G^3+5G^2+10G$
	$(1,1,1)$	1	$-15G^4+15G^2+10G-10$
	$(0,0,0,0)$	0	$-5G^4+5$
	$(0,0,0,0,0)$	0	$-25G^4+25$
	$(0,0,0,0,1)$	0	$-25G^4+25G$
	$(0,0,0,1,1)$	0	$-25G^4+25G^2+25G-25$

Table 2: Witten index $I^{\mathrm{id}\,\sigma}$

tensor product boundary conditions, this expression can be written in terms of a $(k+2)\times(k+2)$ shift matrix $G_{\mu'\mu}=\delta^{(k+2)}_{\mu-\mu'+1,0}$ as

$$I^{\mathrm{id}\,\sigma}(L'_i=0,L)_{\mu'\mu}=(k+2)^{[\frac{n-1}{2}]}(1-G^{-L-1})_{\mu'\mu}. \tag{3.42}$$

This can of course be generalised to permutations consisting of N cycles of length n_ν and labels L_ν. The Witten indices for open strings between such a brane and a $L'_i=0$ tensor product brane are encoded in the matrix

$$I^{\mathrm{id}\sigma}=\prod_{\nu=1}^{N}\,(k+2)^{\left[\frac{n_\nu-1}{2}\right]}\,(1-G^{-L_\nu-1})\ . \tag{3.43}$$

All the Witten indices can be written in terms of G, but the expressions for $I^{\mathrm{id}\,\sigma}$ with arbitrary L'_i and those for $I^{\sigma\,\sigma}$ appear to be more involved than (3.42); see [22] for some results in the quintic case. Nevertheless, even in the absence of a closed formula one can extract each index in a straightforward manner from the topological partition functions determined in section 3.3. In table 1 we list some $I^{\sigma\,\sigma}$ for $k=1,2,3$ and various values of n, L', L, $m=T+\frac{1}{2}(L-M)$ and $\Delta m=m-m'$ (the latter being defined for even n only), and in table 2 some $I^{\mathrm{id}\,\sigma}$ for $k=1,2,3$ and various n, L'_i and L.

4. Permutation branes and linear matrix factorisations

A tensor product $\mathcal{M}_k^{\otimes n}$ of $N=2$ minimal models describes the critical behaviour of a Landau-Ginzburg model with superpotential $W=x_1^d+\ldots+x_n^d$ for $d=k+2$, on a world-

JHEP01(2006)087

sheet without or with boundary. Therefore, one can expect that the CFT B-type branes from above have some counterpart in the form of an LG boundary condition — more concretely that the topological information of the CFT brane can be encoded in a matrix factorisation of the LG potential W. Our proposal is that topological permutation branes correspond to certain *linear matrix factorisations.*

4.1 Linear matrix factorisations

A linear matrix factorisation [26] of a homogeneous polynomial W of degree d in the variables $x_1, \ldots, x_n$ is given by a set of d square matrices $\alpha_0, \ldots, \alpha_{d-1}$ over $\mathbb{C}[x_1, \ldots, x_n]$ all of which are linear in the x_i and satisfy

$$\alpha_0\, \alpha_1 \cdots \alpha_{d-1} = W\, \mathbf{1}\,. \tag{4.1}$$

From the α_i, we can obtain two-factor matrix factorisations by choosing

$$p_0 = \alpha_{\pi(0)} \cdots \alpha_{\pi(\ell-1)} \quad \text{and} \quad p_1 = \alpha_{\pi(\ell)} \cdots \alpha_{\pi(d-1)} \tag{4.2}$$

for $0 < \ell < d-1$ and any cyclic permutation π of $(0, \ldots, d-1)$.

A special class of linear matrix factorisations of $W = \sum x_i^d$ have been constructed explicitly by Backelin, Herzog and Sanders in [26]. For all homogeneous polynomials there exists a unique (up to equivalence and cyclic permutation of the factors) indecomposable linear matrix factorisation with the property

$$\alpha_t(x_i)\alpha_{t+1}(x_j) = \xi \alpha_t(x_j)\alpha_{t+1}(x_i) \qquad i > j \tag{4.3}$$

where $\alpha_t(x_i)$ is the matrix obtained from $\alpha_t(x_1, \ldots, x_n)$ by setting $x_j = 0$ for $j \neq i$, and ξ is a primitive dth root of unity.

In the case $W = x_1^d + \ldots x_n^d$, these factorisations consist of $d^\gamma \times d^\gamma$ matrices, $\gamma = \left[\frac{n-1}{2}\right]$, which can be written as

$$\alpha_i = x_1 + \xi^i\, \alpha_{d,n}, \tag{4.4}$$

where the $\alpha_{d,n}$ can be defined by a recursion formula as follows: One introduces $d \times d$ matrices

$$(\epsilon_1)_{ij} = \xi^{i-1}\, \delta_{i,j-1}\,, \quad (\epsilon_2)_{ij} = \xi^{i-1}\, \delta_{i,j}\,, \quad (\epsilon_3)_{ij} = \delta_{i,j-1}\,, \tag{4.5}$$

where all Kronecker deltas are understood modulo d, as well as the number

$$\mu_n = \begin{cases} 1 & d \text{ odd} \\ \eta & d \text{ even and } \left[\frac{n-1}{2}\right] \text{ even} \\ \eta^{-1} & d \text{ even and } \left[\frac{n-1}{2}\right] \text{ odd} \end{cases}, \tag{4.6}$$

η being a primitive dth root of -1 with $\eta^2 = \xi$. Using these, one defines

$$\alpha_{d,1} = 0, \qquad \alpha_{d,2} = \mu_2\, x_2, \tag{4.7}$$

$$\alpha_{d,n+2} = \epsilon_2 \otimes \alpha_{d,n} + \epsilon_3 \otimes \mu_{n+2}\, x_{n+1}\, \mathbf{1} + \epsilon_1 \otimes x_{n+2}\, \mathbf{1}, \tag{4.8}$$

where the $\mathbf{1}$'s stand for identity matrices of the same size as $\alpha_{d,n}$.

These special linear matrix factorisations have certain nice properties. They are homogeneous in the x_i and indecomposable (i.e. not equivalent to direct sums). Moreover, it is obvious from (4.4) that all the α_i commute, which in particular means that they give rise to matrix factorisations (4.2) not only for π cyclic but for all permutations $\pi \in S_d$. Thus, for every proper subset $I \subset \{0, \ldots, d-1\}$ we obtain the two-factor matrix factorisations

$$M_{I,I^c} = \left(p_0 = \prod_{i \in I} \alpha_i, \; p_1 = \prod_{i \in I^c} \alpha_i \right) , \tag{4.9}$$

where $I^c = \{0, \ldots, d-1\} \backslash I$. For every $\ell = |I|$ these are $\binom{d}{\ell}$ ones.

Note, however, that not all of them have to be inequivalent. To determine, for given number of variables n, the possible equivalences (as defined in (2.4)) between them, first note that since the p_i above are homogeneous, we can restrict to constant matrices U_n, V_n in eq. (2.4), and that two matrix factorisations associated to index sets I and I' as above can be equivalent only if they are of the same degree, i.e. if $|I| = |I'|$. The specific form $p_0 = x_1^{|I|} + \ldots$ enforces $U_n = V_n$, and exploiting (4.4) further one finds that $U_n^{-1} \alpha_{d,n} U_n = \xi^i \alpha_{d,n}$ has to hold for some integer i. Conjugation of a matrix factorisation (4.9) with such a U_n then just shifts the set I to $I + i$ (understood modulo d).

To proceed, one observes that given U_n for a fixed n, one obtains a matrix U_{n+2} conjugating $\alpha_{d,n+2}$ to $\xi^i \alpha_{d,n+2}$ by setting $U_{n+2} = \epsilon_2^i \otimes U_n$. Vice versa, using the explicit form of the matrices ϵ_m and inspecting the recursion relation (4.8) for $\alpha_{d,n+2}$, one can show that any U_{n+2} with the correct conjugation property can be formed from a U_n in this way.

This allows us to list the classes of inequivalent matrix factorisations of the type (4.9): For *odd* $n > 1$, one constructs possible equivalences U_n starting from $U_1 = 1$, which obviously conjugates $\alpha_{d,1} = 0$ to $\xi^i \alpha_{d,1}$. Therefore, in this case matrix factorisations (4.9) defined by the sets I and I' are equivalent if and only if I' is a shift of I. On the other hand, since $\alpha_{d,2}$ is a non-zero rank-1 matrix, there is no matrix U_2 to non-trivially conjugate it. Therefore, for *even* n all the factorisations (4.9) are inequivalent.

The case $n = 1$ provides the simplest example of linear factorisations, where $\alpha_i = x_1$ for all i and we obtain the well-known $d-2$ inequivalent factorisations of minimal model potential $W = x_1^d$, by grouping together ℓ factors x_1 into p_0 and the remaining $d - \ell$ into p_1. For simplicity, they will be denoted $M_\ell(x_1)$ in the following.

The next example $n = 2$ is a little more interesting. Here, the linear matrices are given by $\alpha_i = x_1 + \mu_2 \xi^i x_2$, so that we obtain a matrix factorisation with $p_0 = \prod_{i \in I}(x_1 + \mu_2 \xi^i x_2)$ and $p_1 = \prod_{i \in I^c}(x_1 + \mu_2 \xi^i x_2)$ for every proper subset $I \subset \{0, \ldots, d-1\}$. These factorisations were introduced into the discussion of B-branes in LG models in [40] and then related to CFT permutation branes with $\sigma = (1\,2)$ in [25][3].

For $n > 2$, the factorisations are much harder to treat 'by hand' since the size $d^{[\frac{n-1}{2}]}$ of the matrices grows exponentially with n, which is why later on we will partly resort to computer algebra programmes to perform some of the computations. Note that factorisations (4.9) for the case $n = 3$ and $d = 3$ have already occurred in the classification of

[3]Note however, that our factorisations differ from the ones used in [40, 25] by a shift $x_2 \mapsto \xi^{[\frac{d+1}{2}]} x_2$.

maximal Cohen-Macaulay modules over the cone of the elliptic curve in [41, 42] and in the discussion of D-branes on the elliptic curve in [31, 43].

Ultimately, we are interested in graded matrix factorisations

$$(P_1,\rho_1) \underset{p_0}{\overset{p_1}{\rightleftarrows}} (P_0,\rho_0)\,, \tag{4.10}$$

where apart from the matrix factorisation itself, $\mathbb{Z}_d$-representations ρ_i on the P_i are specified, which are compatible with the module structure (recall that R is graded) and the maps p_i.

For indecomposable matrix factorisations as in (4.9), there is only a choice of one irreducible representation α of $\mathbb{Z}_d$, which determines ρ_1, ρ_0 completely, and we specify it by setting the degree of the element $1 \in R \subset P_0 = R^{d^\gamma}$ to be $\alpha \in \mathbb{Z}_{k+2}$. We denote the corresponding graded factorisations M^α.

4.2 Relation to permutation branes

We would now like to compare these linear matrix factorisations to the boundary states constructed in the previous sections. This will be done by analysing the open topological string sectors on the matrix factorisation side, i.e. the graded Ext-groups $\mathrm{Ext}_R(P,Q)$ between R-modules $P = \mathrm{coker}\, p_1$, $Q = \mathrm{coker}\, q_1$ corresponding to matrix factorisations (p_0,p_1), (q_0,q_1), and comparing them to the respective CFT-results obtained in section 3.3. Here, we will consider the cases where these matrix factorisations are linear factorisations in n variables or tensor products $M^{\otimes}_{L_1,\ldots,L_n} := M_{L_1}(x_1) \otimes \ldots \otimes M_{L_n}(x_n)$ of linear matrix factorisations in one variable, *cf.* (2.5). The generalisation to tensor products of multi-variable linear matrix factorisations is straightforward.

The $\mathbb{Z}_d$-representation of the linear factorisations $M_L(x)$ is specified by $\alpha = \rho_0$, and the $\mathbb{Z}_d$-representations of $M^{\alpha_1}_{L_1}(x_1) \otimes \ldots \otimes M^{\alpha_n}_{L_n}(x_n)$ only depends on $\sum_i \alpha_i$. We define $M^{\otimes\,\alpha}_{L_1,\ldots,L_n}$ to be this tensor product factorisation for an arbitrary partition $\alpha = \sum_i \alpha_i$. These tensor product matrix factorisations reproduce the topological spectra of tensor product boundary states (3.10) (a discussion of this can be found in [24]), the precise correspondence being

$$\| L_1,\ldots,L_n,S_1,\ldots,S_n,M = \textstyle\sum_i L_i - 2\alpha\rangle\!\rangle^{\mathrm{id}}_B \quad\longmapsto\quad M^{\otimes\,\alpha}_{L_1,\ldots,L_n}\,. \tag{4.11}$$

We propose the following *correspondence between CFT permutation boundary states and matrix factorisations*:

$$\begin{array}{lll} n \text{ odd}: & \|{\scriptstyle L,L-2\alpha,S_1=0,\ldots,S_n=0}\rangle\!\rangle^{\sigma}_B & \longmapsto \quad M^{\alpha}_{\{0,\ldots,L\},\{L+1,\ldots,d-1\}}\,, \\ n \text{ even}: & \|{\scriptstyle L,L-2\alpha,T=m-\alpha,S_1=0,\ldots,S_n=0}\rangle\!\rangle^{\sigma}_B & \longmapsto \; M^{\alpha}_{\{0,\ldots,L\}-m,\{L+1,\ldots,d-1\}-m}\,, \end{array} \tag{4.12}$$

where we use the notations of (4.9), and where elements in the sets I are understood to be taken modulo $d = k+2$.

Note that, for odd n, factorisations M_{I,I^c} and M_{I+i,I^c+i} are equivalent, whereas for even n this is not the case and the respective shift m is determined by the boundary state labels (L,M,T) through $m = T + \frac{1}{2}(L-M)$.

For the case $n = 1$, a single minimal model, this correspondence is of course well known. We have spelled out the topological spectra in section 3.3, and the Ext-groups of the corresponding matrix factorisations can easily be calculated, see e.g. [17, 23, 24].

For the next complicated case $n = 2$, the linear matrix factorisations still have rank one, so the correspondence can still be checked by hand in a straightforward way. Ext-groups involving two $n = 2$ linear factorisations or one linear and one tensor product factorisation were first studied in [24]. The comparison with CFT permutation boundary states for $\sigma = (1\,2)$ has been carried out in great detail in the recent work [25], so we refrain from repeating the calculations here.

Whenever $n > 2$, the linear matrix factorisations involve higher rank matrices, and the computation of the Ext-groups may become quite tedious. We do not yet have a general derivation for all possible combinations of linear and tensor product factorisations. The Ext-groups between M^{α}_{I,I^c} and $M^{\otimes,\beta}_{L_1,L_2=0,\ldots,L_n=0}$ are calculated for arbitrary n and d in section 4.3, exploiting certain constructions from homological algebra. The results are in agreement with the correspondence proposed above.

To check agreement also for the other spectra (in particular the ones involving two higher rank linear matrix factorisations), we resort to calculating the respective Ext-groups on a case-by-case basis on the computer. For this purpose we used the computer algebra program Macaulay2 [27]. Some of the results of these calculations are presented in section 4.4 below. All tests show agreement with the CFT results obtained in section 3 and confirm the correspondence (4.12).

Before we turn to Ext-groups, we can apply a simpler test to our correspondence between linear matrix factorisations and boundary states, concerning the behaviour under the charge symmetry $\mathbb{Z}_d^n$, whose generators act as $g_i\ :\ x_j \mapsto \xi^{\delta_{ij}}\, x_j$ on the LG variables and multiply CFT Ishibashi states by $\xi^{\frac{1}{2}(m_i+\overline{m}_i)}$. From formulae (3.17), (3.23) for the permutation boundary states, one sees that each g_i shifts the T-label by $(-1)^i$ for n even, while it leaves the boundary state labels invariant when n is odd. For the linear matrix factorisations, on the other hand, g_i induces a shift $I \mapsto I + (-1)^i$ of the index set — which is an equivalence for n odd, but changes the equivalence class of the matrix factorisation for even n, in accordance with the proposed correspondence.

4.3 Calculation of some Ext-groups

Let $M^{\otimes}_{L_1,\ldots,L_n} = (q_0, q_1)$ be a tensor product matrix factorisation as above and $M_{I,I^c} = (p_0, p_1)$ with $|I| = L + 1$ any linear matrix factorisation of degree $L + 1$. In this section, we aim at calculating the groups $\mathrm{Ext}_R(\mathrm{coker}\, q_1, \mathrm{coker}\, p_1)$. To do this, we use a relation between the factorisation (q_0, q_1) and the module $N = N_{L_1,\ldots,L_n} := R/(x_1^{L_1+1}, \ldots, x_n^{L_n+1})$. One way to establish this connection, namely by deconstructing the tensor product matrix factorisation (q_0, q_1), is presented in appendix A. Here, we will take a more direct approach and construct a free resolution of N which becomes 2-periodic after $(n - 1)$ steps with periodic part given by (q_0, q_1). In fact, this is a special case of a more general construction due to Eisenbud [34]. For a commutative ring A and an ideal $\mathcal{I}$, Eisenbud constructs a free resolution of a $B = A/\mathcal{I}$-module V out of an A-free resolution of V.

In our case $A = \mathbb{C}[x_1, \ldots, x_n]$, $\mathcal{I} = (W = \sum_i x_i^d)$ and $B = R$. As A-free resolution of N we use the *Koszul complex* of $(x_1^{L_1+1}, \ldots, x_n^{L_n+1})$ which is a minimal A-free resolution of N of length $(n+1)$

$$0 \longrightarrow K_n \xrightarrow{\delta} K_{n-1} \xrightarrow{\delta} \ldots \xrightarrow{\delta} K_1 \xrightarrow{\delta} K_0 \longrightarrow N \longrightarrow 0\,, \tag{4.13}$$

where $K_i = \Lambda^i V$ is the i^{th} exterior power of $V = A^n$ with A-basis $\{e_1, \ldots, e_n\}$ and co-differential

$$\delta\, e_{i_1} \wedge \cdots \wedge e_{i_p} := \sum_{j=1}^{p} (-1)^{j-1} x_{i_j}^{L_{i_j}+1} e_{i_1} \wedge \cdots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \cdots \wedge e_{i_p}\,. \tag{4.14}$$

To obtain an R-free resolution of N from this, one first introduces the maps $\sigma : K_i \longrightarrow K_{i+1}$ defined by

$$\sigma : \omega \longmapsto \left(\sum_i x_i^{d-L_i-1} e_i \right) \wedge \omega\ , \tag{4.15}$$

which satisfy $\delta\sigma + \sigma\delta = W$. Furthermore let $T_i := A\, t^i$ for $i \geq 0$ and define the operator $\lambda : T_n \longrightarrow T_{n-1}$ by $\lambda(t^n) = t^{n-1}$ for $n \geq 1$ (and $\lambda := 0$ on T_0). Then we obtain the chain of A-modules

$$\ldots \longrightarrow F_l \xrightarrow{\widetilde{\delta}} F_{l-1} \xrightarrow{\widetilde{\delta}} \ldots \xrightarrow{\widetilde{\delta}} F_1 \xrightarrow{\widetilde{\delta}} F_0\,, \tag{4.16}$$

with

$$F_i = \bigoplus_{j=0}^{[\frac{i}{2}]} K_{i-2j} \otimes T_j\,, \qquad \widetilde{\delta} = \delta \otimes \mathrm{id} + \sigma \otimes \lambda\,. \tag{4.17}$$

One has $(\widetilde{\delta})^2 = W \otimes t$, and since $F_0 = K_0$, $F_1 = K_1$, we have $F_0/\operatorname{im}(\widetilde{\delta}) = N$. Therefore, tensoring the complex (4.16) with R, we obtain an R-free resolution $\ldots \to \widetilde{F}_i \to \widetilde{F}_{i-1} \to \ldots \to \widetilde{F}_0 \to N \to 0$ of N with $\widetilde{F}_i = F_i \otimes R$. By construction this complex is 2-periodic from position $i = n$.

Since the T-factors in the periodic part are redundant, the latter can be represented as follows

$$\Phi_j = \bigoplus_i \Lambda^{n-2i+j} V\ ,\ j \in \{0,1\}\,, \qquad \widehat{\delta} = \delta + \sigma : \Phi_j \longrightarrow \Phi_{j+1}\,. \tag{4.18}$$

Again $(\widehat{\delta})^2 = W$, and $\widetilde{\Phi}_i = \Phi_i \otimes R$, together with maps induced by $\widehat{\delta}$, is the periodic part of the R-free resolution $\widetilde{F}_i$ of N.

Now we claim that this periodic part is isomorphic to the tensor product matrix factorisation

$$M^{\otimes}_{L_1,\ldots,L_n} = \left(Q_1 \underset{q_0}{\overset{q_1}{\rightleftarrows}} Q_0 \right)\,. \tag{4.19}$$

This can easily be shown by induction on n: Let $A' = \mathbb{C}[x_1, \ldots, x_{n-1}]$, $\Phi_i{}'$ and $\widehat{\delta}'$ be defined as above for the situation with $(n-1)$ variables $(x_1, \ldots, x_{n-1})$ and $A'' = \mathbb{C}[x_n]$, $\Phi_i{}''$ and $\widehat{\delta}''$ be defined as above for the situation with one variable x_n. Then Φ is given by the tensor product of Φ' and Φ'': $\Phi_i \cong \bigoplus_{r+s+i \in 2\mathbb{Z}} \Phi'_r \otimes_A \Phi''_s$ and $\widehat{\delta} = \widehat{\delta}' \otimes \mathrm{id} + \mathrm{id} \otimes \widehat{\delta}''$. Thus, if $(\Phi'_i, \widehat{\delta}')$

and $(\Phi''_i, \widehat{\delta}'')$ are isomorphic to the respective matrix factorisations, so is $(\Phi_i, \widehat{\delta})$. Therefore we only have to show the statement for the case of one variable, where it is obvious:

$$\begin{aligned} &\Phi_0 \cong \Lambda^1 A e_1 \cong \mathbb{C}[x]\,, \quad \Phi_1 \cong \Lambda^0 A e_1 \cong \mathbb{C}[x]\,, \\ &\delta = x^{L+1} : \Phi_0 \to \Phi_1\,, \quad \sigma = x^{d-L-1} : \Phi_1 \to \Phi_0\,. \end{aligned} \tag{4.20}$$

In particular, for a single variable we have $Q_i \cong \widetilde{\Phi}_i$. This proves that the periodic part $\widetilde{\Phi}_i$ of the R-free resolution $\widetilde{F}_i$ of N is given by the respective tensor product matrix factorisation $M^{\otimes}_{L_1,\dots,L_n}$. However, we need to treat the modules as graded ones, and there is a shift of the $\mathbb{Z}_d$-grading between Q_i and $\widetilde{\Phi}_i$: Let us assume that the gradings α of the matrix factorisations are zero. Then, in the one variable case Q_0 has degree 0, whereas $\widetilde{\Phi}_0$ has degree $L_1 + 1$ (since this is the degree of the basis vector e_1, due to δ in (4.14) having degree 0). Thus, taking the degrees into account, we find that the tensor product matrix factorisation is isomorphic to the periodic part of the resolution of $N(-\sum_i(L_i+1))$, N with degree shifted by $-\sum_i(L_i+1)$.[4] Let us for the moment abbreviate $\sum_i(L_i+1) =: \mu$. Then we have

$$\mathrm{Ext}^i_R(\mathrm{coker}\, q_1, M) \cong \mathrm{Ext}^{i+n}_R(N, M)(\mu) \tag{4.21}$$

for all $i > 0$ and all R-modules M.

To calculate the right hand side of (4.21) for $M = \mathrm{coker}\, p_1$, we use the following fact (see e.g. Lemma 3.1.16 in [44]): Let S be a graded ring, U and V be S-modules, and $x \in S$ a homogeneous element that annihilates U and is S- and V-regular[5]. Then one has

$$\mathrm{Ext}^{i+1}_S(U, V) \cong \mathrm{Ext}^i_{S/(x)}(U, V/xV)(-\deg(x))\,. \tag{4.22}$$

Noting that $(x_2^{L_2+1}, \dots, x_n^{L_n+1})$ is an R- and $\mathrm{coker}\, p_1$-regular sequence[6] in the annihilator of N, we obtain

$$\begin{aligned} &\mathrm{Ext}^i_R(\mathrm{coker}\, q_1, \mathrm{coker}\, p_1) \cong \mathrm{Ext}^{i+n}_R\big(N, \mathrm{coker}\, p_1\big)\big(\textstyle\sum_i(L_i+1)\big) \\ &\quad \cong \mathrm{Ext}^{i+1}_{R/(x_2^{L_2+1},\dots,x_n^{L_n+1})}\big(N, \mathrm{coker}\, p_1/(x_2^{L_2+1},\dots,x_n^{L_n+1})\,\mathrm{coker}\, p_1\big)(L_1+1)\,. \end{aligned} \tag{4.23}$$

The right hand side is easy to determine in the case $L_2 = \ldots = L_n = 0$, in which

$$\begin{aligned} &R/(x_2,\dots,x_n) \cong \mathbb{C}[x_1]/(x_1^d) =: S\,, \\ &\mathrm{coker}\, p_1/(x_2,\dots,x_n)\,\mathrm{coker}\, p_1 \cong \Big(S/x_1^{d-L-1}S\Big)^{d^\gamma}\,, \\ &N \cong S/x_1^{L_1+1}S\,, \end{aligned} \tag{4.24}$$

and N has the obvious S-free resolution

$$\ldots \longrightarrow S \xrightarrow{x_1^{L_1+1}} S \xrightarrow{x_1^{d-L_1-1}} S \xrightarrow{x_1^{L_1+1}} S \longrightarrow \mathbb{C}[x_1]/(x_1^{L_1+1}) \longrightarrow 0\,, \tag{4.25}$$

[4] If $M = \bigoplus_n M_n$ is a graded module and μ an integer, $M(\mu)$ is the module with $M(\mu)_n := M_{n-\mu}$. (This is not be confused with shifted complexes, usually denoted $C[\mu]$.) Shifting the degree of a module also affects the degree of its Ext-groups, namely $\mathrm{Ext}(M(\mu), N) = \mathrm{Ext}(M, N)(-\mu)$.

[5] An element $x \in S$ is V-regular if $xv = 0$ for $v \in V$ implies $v = 0$.

[6] For an R-module M, an M-regular sequence is a sequence $(a_1, \dots, a_n)$ in R such that a_1 is M-regular and a_{j+1} is $(M/(a_1,\dots,a_j)M)$-regular for all $1 \le j \le n$.

JHEP01(2006)087

which can be used to obtain the respective Ext-groups

$$\begin{aligned}\mathrm{Ext}^{2i+1}_S(N, S/x_1^{d-L-1}S) &\cong x_1^{\max(0,L_1-L)}\mathbb{C}[x_1]/(x_1^{\min(d-L-1,L_1+1)})(-L_1-1)\,,\\ \mathrm{Ext}^{2i}_S(N, S/x_1^{d-L-1}S) &\cong x_1^{\max(0,d-L-L_1-2)}\mathbb{C}[x_1]/(x_1^{\min(d-L-1,d-L_1-1)})\,,\end{aligned}$$

for $i > 0$. Putting everything together, we obtain

$$\begin{aligned}&\mathrm{Ext}^{2i}_R(\operatorname{coker} q_1, \operatorname{coker} p_1) \cong d^\gamma\left(x_1^{\max(0,L_1-L)}\mathbb{C}[x_1]/(x_1^{\min(d-L-1,L_1+1)})\right)\,,\\ &\mathrm{Ext}^{2i+1}_R(\operatorname{coker} q_1, \operatorname{coker} p_1)\\ &\qquad\qquad \cong d^\gamma\left(x_1^{\max(0,d-L-L_1-2)}\mathbb{C}[x_1]/(x_1^{\min(d-L-1,d-L_1-1)})(L_1+1)\right)\,.\end{aligned}$$

This agrees, via the correspondence (4.11), (4.12), precisely with the topological spectra (3.40), (3.41), and in particular yields the correct Witten index (3.42).

For arbitrary $L_2, \ldots, L_n$, the computation of the right hand side of (4.23) is more involved. Case-by-case checks using Macaulay2 however show agreement in these cases as well.

4.4 Computer checks

As mentioned above, we have not yet been able to construct a rigorous proof for the general correspondence (4.12). Therefore, we collect additional evidence for it based on case-by-case calculations of the respective Ext-groups, using the computer algebra program Macaulay2 [27].

Macaulay2 does exact calculations using rings which may be of the form $K[x_1, \ldots, x_n]/\mathcal{I}$, where $\mathcal{I}$ is an ideal and K some field, which we define to be the field extension $\mathbb{Q}(a)$, where a is a fundamental root of 1 if d is odd and of -1 if d is even. This is done by setting $K = \mathbb{Q}[a]/(f(a))$ for the appropriate polynomial f.

Macaulay2 has a built-in procedure to calculate the Ext-groups. The $\mathbb{Z}_d$-representations, which correspond to the degrees of the graded modules P_i, Q_i and the maps between them, are also calculated by Macaulay2.

Although we may use Macaulay2 to calculate the full algebra of the chiral rings, for brevity we only present the calculation of the index $I(P,Q) = \dim \mathrm{Ext}^2_R(P,Q) - \dim \mathrm{Ext}^1_R(P,Q)$ here. Our code can be found in appendix B.1. It calculates $I(P,Q)$ for two given graded matrix factorisations P and Q and expresses it in terms of the shift matrix $G_{\mu'\mu}$, where here $\mu' = \alpha'$ and $\mu = \alpha$ specify the $\mathbb{Z}_d$-representations of P and Q respectively. A few results are displayed in appendix B.2.

All our tests showed agreement of the topological spectra of permutation boundary conditions on the one hand and the graded Ext-groups of the matrix factorisations corresponding to them via (4.11), (4.12) on the other. This is in particular the case for the examples listed in tables 1 and 2.

Acknowledgments

We would like to thank M. Baumgartl, I. Brunner, G. D'Appollonio, G. Ellingsrud, M. Gaberdiel, R. Ile, W. Lerche, C.A. Lütken and P. West for useful discussions. This work has been partly supported by a EC Marie Curie Training Site Fellowship under contract no. HPMT-CT-2001-00296, by the EC network "EUCLID" under contract no. HPRN-CT-2002-00325, and by the PPARC grant PPA/G/O/2002/00475. We thank the Erwin-Schrödinger Institut, Vienna, for hospitality.

A. Deconstructing tensor product factorisations

Here we would like to give a slightly different derivation of (4.23). As in section 4.3, we take $M^{\otimes}_{L_1,\dots,L_n} = (q_0,\, q_1)$ to be a tensor product factorisation and $M_{I,I^c} = (p_0,\, p_1)$ to be any linear matrix factorisation of degree $|I| = L+1$. To calculate the modules $\mathrm{Ext}_R(\mathrm{coker}\, q_1, \mathrm{coker}\, p_1)$, we can make use of the tensor product structure of $M^{\otimes}_{L_1,\dots,L_n}$. Namely,

$$q_0 = \begin{pmatrix} q'_0 \otimes 1 & -1 \otimes q''_1 \\ 1 \otimes q''_0 & q'_1 \otimes 1 \end{pmatrix}, \quad q_1 = \begin{pmatrix} q'_1 \otimes 1 & 1 \otimes q''_1 \\ -1 \otimes q''_0 & q'_0 \otimes 1 \end{pmatrix}, \tag{A.1}$$

where

$$\left(Q'_1 \underset{q'_0}{\overset{q'_1}{\rightleftarrows}} Q'_0 \right) = M^{\otimes}_{L_1,\dots,L_{n-1}} \tag{A.2}$$

is the tensor product factorisation of $W'(x_1,\dots,x_{n-1}) = x_1^d + \dots + x_{n-1}^d$ and $(q''_0 = x_n^{L_n+1}, q''_1 = x^{d-L_n-1})$ the factorisation of $W''(x_n) = x_n^d$. The $Q'_i \otimes Q''_j$ are free, and the long exact Ext-sequence obtained from

$$0 \longrightarrow Q'_1 \otimes Q''_0 \longrightarrow \mathrm{coker}\, q_0 \longrightarrow \mathrm{coker}\left(\mathrm{id}_{Q'_0} \otimes x_n^{L_n+1},\; q'_1 \otimes \mathrm{id}_{Q''_1} \right) \longrightarrow 0 \tag{A.3}$$

gives rise to the following isomorphisms

$$\begin{aligned} \mathrm{Ext}^i_R(\mathrm{coker}\, q_1, \cdot) &\cong \mathrm{Ext}^{i+1}_R(\mathrm{coker}\, q_0, \cdot) \\ &\cong \mathrm{Ext}^{i+1}_R\big(\mathrm{coker}\; q'_1 / x_n^{L_n+1}\, \mathrm{coker}\; q'_1, \cdot \big)\,. \end{aligned} \tag{A.4}$$

Following the degrees in all the steps, one sees that the degree of the third Ext in (A.4) is shifted relative to the one of the Ext on the left hand side by $L_n + 1$. As in section 4.3, we use the fact (see e.g. Lemma 3.1.16 in [44]) that for any ring S and any S-modules U and V with a homogeneous $x \in S$ that annihilates U and is R- and V-regular

$$\mathrm{Ext}^{i+1}_S(U, V) \cong \mathrm{Ext}^i_{S/(x)}(U, V/xV)(-\deg(x))\,. \tag{A.5}$$

Since $x_n^{L_n+1}$ is $\mathrm{coker}\, p_1$-regular this gives

$$\begin{aligned} &\mathrm{Ext}^i_R(\mathrm{coker}\, q_1, \mathrm{coker}\, p_1) \\ &\cong \mathrm{Ext}^i_{R/(x_n^{L_n+1})} \left(\mathrm{coker}\, q'_1 /\, x_n^{L_n+1}\, \mathrm{coker}\, q'_1\,,\; \mathrm{coker}\, p_1 /\, x_n^{L_n+1}\, \mathrm{coker}\, p_1 \right)\,. \end{aligned} \tag{A.6}$$

Furthermore, $\mathrm{coker}\, q'_1 /\, x_n^{L_n+1}\, \mathrm{coker}\, q'_1 \cong \mathrm{coker}\, \hat{q}'_1$, where $\hat{q}'_i$ are the induced maps between the $Q_i / x_n^{L_n+1} Q_i$, which again have tensor product form (A.1). Because $(x_2^{L_2+1}, \dots, x_n^{L_n+1})$

is an R- and $\operatorname{coker} p_1$-regular sequence, we therefore obtain (4.23) inductively:

$$\begin{aligned}
&\operatorname{Ext}^i_R(\operatorname{coker} q_1, \operatorname{coker} p_1)\\
&\quad\cong \operatorname{Ext}^i_{R/(x_2^{L_2+1},\ldots,x_n^{L_n+1})}\left(\widetilde{N},\ \operatorname{coker} p_1 \big/ (x_2^{L_2+1},\ldots,x_n^{L_n+1}) \operatorname{coker} p_1\right)\\
&\quad\cong \operatorname{Ext}^{i+1}_{R/(x_2^{L_2+1},\ldots,x_n^{L_n+1})}\left(N,\ \operatorname{coker} p_1 \big/ (x_2^{L_2+1},\ldots,x_n^{L_n+1}) \operatorname{coker} p_1\right)(L_1+1)
\end{aligned}$$

where $N = R/(x_1^{L_1+1},\ldots,x_n^{L_n+1})$ as in section 4.3, and where we have used $\widetilde{N} := R/(x_1^{d-L_1-1}, x_2^{L_2+1},\ldots,x_n^{L_n+1})$. This provides an alternative derivation of (4.23).

B. Calculations with Macaulay2

B.1 Code

The procedure `init` sets up the rings necessary for dealing with linear matrix factorisations of $W = x_1^d + \ldots + x_n^d$.

```
-- Sets up necessary fields, rings.
init = (d,n) -> (
     KK=QQ[G]/(1-G^d);
     toField KK;
     K=QQ[a]/((factors(1+(-a)^d))_0);
     toField K;
     K.isHomogeneous=true;
     A=K[x_1 .. x_n];
     f=sum apply(toList(x_1 .. x_n),y->y^d);
     R=A/f;);
```

The procedure `linmf` creates linear matrix factorisations (4.9), where the first argument is an ordered set of indices labelling the variables used in the factorisation, and the second one is the set I defining it. For this procedure we also need some other functions.

```
-- Function subracts two sets.
subt = (I1,I2) -> (
     I1=I2|I1;
     I1=unique(I1);
     I1=drop(I1,#I2);
     return(I1));

-- Matrices \epsilon_1, \epsilon_2, \epsilon_3
e3mat = (R,d) -> (return(map(R^d,R^d,
                         (i,j)->(if j==(i+1)%d then 1 else 0))));
e1mat = (R,d,v) -> (return(map(R^d,R^d,
                           (i,j)->(if j==(i+1)%d then v^i else 0))));
e2mat = (R,d,v) -> (return(map(R^d,R^d,
                           (i,j)->(if i==j then v^i else 0))));

-- Inductively constructs \alpha matrices
nplustwo = (R,a,I,n,alpha) -> (
     d:=degree R;
     mu:=(if (even ((n-1)//2)) then a else a^(-1));
```

```
     if even d then
       (R_(I_(n+1))*e1mat(R,d,a^2))**id_(source alpha)
           +(mu*R_(I_n)*e3mat(R,d))**id_(source alpha)
           +e2mat(R,d,a^2)**alpha
     else
       (R_(I_(n+1))*e1mat(R,d,a))**id_(source alpha)
           +(R_(I_n)*e3mat(R,d))**id_(source alpha)
           +e2mat(R,d,a)**alpha);

alphan = (R,a,I) -> (
     mu:=(if (even ((#I-1)//2)) then a else a^(-1));
     alpha:= (if (even (#I)) then matrix {{mu*R_(I_1)}}
                             else matrix {{0_R}});
     m:=(if (even (#I)) then 0 else 1);
     for i from (if (even (#I)) then 1 else 0) to floor((#I)/2)-1 do
       alpha=nplustwo(R,a,I,2*i+m,alpha);
     alpha);

-- Create linear mf
linmf = (I,J) -> (
     d:=degree R;
     a:=((coefficientRing R)_0)_R;
     I=apply(I,i->i-1);
     J=apply(J,i->i%d);
     A=alphan(R,a,I);
     N=rank source A;
     b := if even d then
            z -> (R_(I_0)**id_(R^(rank source A)) + a_R^(2*z) * A)
          else
            z -> (R_(I_0)**id_(R^(rank source A)) + a_R^z * A);
     g=map(R^N,R^N**(R^{#J-d}),product apply(subt(toList(0..d-1),J),b));
     f=map(source g, target g,(product apply(J,b)));
     return(f,g));
```

The procedure **tpmf** creates tensor product matrix factorisations. The first argument is again the ordered set of variable indices and the second one the ordered set of the respective L-labels.

```
-- Creates the tensor product of two matrix factorisations
tp = (p,q) -> (
     Rp0=target p_1;Rp1=source p_1;
     Rq0=target q_1;Rq1=source q_1;
     return(
     map(p_0**id_(Rq0)|-id_(Rp1)**q_1)||(id_(Rp0)**q_0|p_1**id_(Rq1)),
     map(p_1**id_(Rq0)|id_(Rp0)**q_1)||(-id_(Rp1)**q_0|p_0**id_(Rq1))));

-- Creates the tensor product of one-variable factorisations
tpmf = (I,J) -> (
     d=degree R;
     if #I==1 then tf=linmf(I,toList(0..J_0))
              else tf=tp(tpmf(drop(I,-1),J),
                         linmf((I_(#I-1)..I_(#I-1)),toList(0..J_(#I-1))));
     return(tf));
```

The procedure deg calculates the bosonic and fermionic partition functions and ind the index $I(P, Q)$ between two matrix factorisations.

```
-- Calculates the degrees of the respective Ext-modules
deg = (n,p,q) -> (
     n=-abs(n)%2+2;
     Mp=coker p_1;
     Mq=coker q_1;
     emod=Ext^n(Mp,Mq);
     e=if (dim emod!=0) then matrix {{}} else super basis emod;
     ed=apply(numgens source e,i->((degree e_i)_0));
     return(sum(ed,i->G^(i))));

-- Calculates the index I
ind = (p,q) -> (
     return(deg(0,p,q)-deg(1,p,q)));
```

B.2 Results

In the following we demonstrate how to use the above code:

```
Macaulay 2, version 0.9.2
--Copyright 1993-2001, D. R. Grayson and M. E. Stillman
--Singular-Factory 1.3c, copyright 1993-2001, G.-M. Greuel, et al.
--Singular-Libfac 0.3.3, copyright 1996-2001, M. Messollen

i1 : load "linmf.m2"
--loaded linmf.m2
```

As an example, for $d = 4$ and $n = 3$, we set up a linear matrix factorisation $M_{\{0\}}$ and a tensor product factorisation $M^{\otimes}_{0,1,2}$ and calculate indices I.

```
i2 : init(4,3)

i3 : p=linmf({1,2,3},{0});

i4 : p_0

o4 = {3} | x_1        ax_2+x_3 0          0        |
     {3} | 0          x_1      ax_2+a2x_3 0        |
     {3} | 0          0        x_1        ax_2-x_3 |
     {3} | ax_2-a2x_3 0        0          x_1      |

             4       4
o4 : Matrix R  <--- R

i5 : q=tpmf({1,2,3},{0,1,1});

i6 : q_0

o6 = {3} | x_1   -x_2^2 -x_3^2 0      |
     {2} | x_2^2 x_1^3  0      -x_3^2 |
     {2} | x_3^2 0      x_1^3  x_2^2  |
     {7} | 0     x_3^2  -x_2^2 x_1    |
```

```
              4       4
o6 : Matrix R  <--- R

i7 : ind(p,p)

         3     2
o7 = - 6G  + 6G  + 2G - 2

o7 : KK

i8 : ind(q,p)

         3     2
o8 = - 4G  + 4G  + 4G - 4

o8 : KK
```

The indices calculated for these matrix factorisations agree via the correspondence (4.11), (4.12) with the respective entries in tables 1, 2.

For even cycle length, e.g. $n = 4$, the factorisations M_I not only depend on the cardinality of I.

```
i9 : init(4,4)

i10 : p=linmf({1,2,3,4},{0});

i11 : q=linmf({1,2,3,4},{1});

i12 : ind(p,p)

        3      2
o12 = 4G  + 10G  + 4G + 2

o12 : KK

i13 : ind(p,q)

          3     2
o13 = - 2G  + 4G  - 2G - 4

o13 : KK
```

Upon comparison with tables 1 and 2 also these results agree with the correspondence (4.12).

C. Linear matrix factorisations and the quintic

For the quintic hypersurface in $\mathbb{P}^4$, the charges of the minimal model tensor product branes were calculated in [5]. The rank of the B-brane charge lattice is equal to $N = \sum_i b^{2i}$, where b^j are the Betti numbers of the underlying manifold; $N = 4$ for the quintic.

Permutation	M	Charges				Chern characters			
		$D6$	$D4$	$D2$	$D0$	rk	ch_1	ch_2	ch_3
(1)(2)(3)(4)(5)	0	1	0	0	0	1	0	0	0
	2	−4	−1	−8	5	−4	1	5/2	5/6
	4	6	3	19	−10	6	−3	−5/2	5/2
	6	−4	−3	−14	10	−4	3	−5/2	−5/2
	8	1	1	3	−5	1	−1	5/2	−5/6
(1)(2)(3)(45)	0	3	2	11	−6	3	−2	0	7/3
	2	−1	−1	−3	4	−1	1	−5/2	−1/6
	4	0	0	0	−1	0	0	0	−1
	6	1	0	0	−1	1	0	0	−1
	8	−3	−1	−8	4	−3	1	5/2	−1/6
(1)(2)(345)	0	0	0	−5	0	0	0	5	0
	2	−5	0	−5	5	−5	0	5	5
	4	10	5	35	−15	10	−5	−15/2	35/6
	6	−5	−5	−20	15	−5	5	−15/2	−35/6
	8	0	0	−5	−5	0	0	5	−5
(1)(23)(45)	0	0	0	−1	0	0	0	1	0
	2	−1	0	−1	1	−1	0	1	1
	4	2	1	7	−3	2	−1	−3/2	7/6
	6	−1	−1	−4	3	−1	1	−3/2	−7/6
	8	0	0	−1	−1	0	0	1	−1
(1)(2345)	0	5	4	22	−10	5	−4	0	20/3
	2	0	−1	2	5	0	1	−15/2	5/6
	4	0	−1	−3	0	0	1	−5/2	−25/6
	6	0	−1	−8	0	0	1	5/2	−25/6
	8	−5	−1	−13	5	−5	1	15/2	5/6
(12)(345)	0	5	4	22	−10	5	−4	0	20/3
	2	0	−1	2	5	0	1	−15/2	5/6
	4	0	−1	−3	0	0	1	−5/2	−25/6
	6	0	−1	−8	0	0	1	5/2	−25/6
	8	−5	−1	−13	5	−5	1	15/2	5/6
(12345)	0	−5	0	−25	0	−5	0	25	0
	2	−5	5	15	0	−5	−5	25/2	125/6
	4	20	10	80	−25	20	−10	−25	50/3
	6	−5	−10	−30	25	−5	10	−25	−50/3
	8	−5	−5	−40	0	−5	5	25/2	−125/6

Table 3: Charges and Chern characters of $L = 0$ permutation branes for the quintic, computed from the Witten index with tensor product branes

Since the charges of the $L_i = 0$ tensor product branes span the whole charge lattice over $\mathbb{Q}$ in this case (they do not provide an integral basis), one can extract the charges of the permutation branes from the Witten index $I^{\mathrm{id}\,\sigma}$ between $L_i = 0$ tensor product branes and the permutation branes determined in (3.43): Let the columns of the matrix Q^{id} contain the charges of the $L_i = 0$ tensor product branes[7], and the "large volume

[7]The charge matrix Q^{id} for a general Gepner model may be determined from the results of [45]

intersection matrix" I in the charge basis be given by $I_{ij} = (-1)^{i+1}\delta_{i,n-i+1}$. As long as Q^{id} has rank N (meaning that the tensor product branes span the whole charge lattice over $\mathbb{Q}$), the charges of σ-permutation branes are given by $Q^\sigma = JI^{\mathrm{id}\,\sigma}$, where $J(Q^{\mathrm{id}})^t I = 1$.

By the method outlined above, we can compute all permutation brane charges from the intersection form $I^{\mathrm{id}\,\sigma}$ alone, for any model where the tensor product branes generate the charge lattice over $\mathbb{Q}$. The resulting charges for all B-type permutation branes with L-labels 0 on the quintic are displayed in table 3. Note that if the charge lattice is generated (over $\mathbb{Q}$) by the tensor product branes, it easily follows from the form of $I^{\mathrm{id}\,\sigma}$ given in (3.43) that the charges of the permutation branes with $L = 0$ generate those of all permutation branes.

Among the branes for the permutation (12)(3)(4)(5), one finds one with charges of a (single) D0-brane. The absence of other charges (D2,4,6) for this boundary state was already noted in [22, 46], where however the normalisation was not discussed. The correct normalisation was first obtained in [25], where it was noticed that the charges of the (12)(3)(4)(5) permutation branes indeed generate the whole charge lattice (over the integers).

Another interesting property to note is that, up to normalisation, the intersection forms (3.43) only depend on the number of cycles of the respective permutation. The normalisation is given by $(k+2)^N$ with $N = \sum_\nu [\frac{n_\nu - 1}{2}]$ depending on the cycle lengths n_ν only. In particular the normalisation for the permutation branes (12)(345) and (1)(2345) and therefore their charges are identical. Only for these permutation branes D4 branes (without D6-brane charge) show up. None of these boundary states, however, is a "pure" D4 brane, instead there is some admixture of D2-charge.

References

[1] M. Kontsevich, *Homological algebra of mirror symmetry*, in *Proceedings of the international congress of mathematicians*, pp. 120-139, Birkhäuser, Berlin 1994.

[2] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[3] D. Gepner, *Space-time supersymmetry in compactified string theory and superconformal models*, *Nucl. Phys.* **B 296** (1988) 757.

[4] A. Recknagel and V. Schomerus, *D-branes in Gepner models*, *Nucl. Phys.* **B 531** (1998) 185 [hep-th/9712186].

[5] I. Brunner, M.R. Douglas, A.E. Lawrence and C. Römelsberger, *D-branes on the quintic*, *JHEP* **08** (2000) 015 [hep-th/9906200].

[6] D.-E. Diaconescu and C. Römelsberger, *D-branes and bundles on elliptic fibrations*, *Nucl. Phys.* **B 574** (2000) 245 [hep-th/9910172].

[7] P. Kaste, W. Lerche, C.A. Lütken and J. Walcher, *D-branes on K3-fibrations*, *Nucl. Phys.* **B 582** (2000) 203 [hep-th/9912147].

[8] E. Scheidegger, *D-branes on some one- and two-parameter Calabi-Yau hypersurfaces*, *JHEP* **04** (2000) 003 [hep-th/9912188].

JHEP01(2006)087

[9] S. Govindarajan, T. Jayaraman and T. Sarkar, *Worldsheet approaches to D-branes on supersymmetric cycles*, *Nucl. Phys.* **B 580** (2000) 519 [hep-th/9907131].

[10] S. Govindarajan and T. Jayaraman, *On the Landau-Ginzburg description of boundary CFTs and special lagrangian submanifolds*, *JHEP* **07** (2000) 016 [hep-th/0003242].

[11] K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry*, hep-th/0005247.

[12] N.P. Warner, *Supersymmetry in boundary integrable models*, *Nucl. Phys.* **B 450** (1995) 663 [hep-th/9506064].

[13] I. Brunner, M. Herbst, W. Lerche and B. Scheuner, *Landau-Ginzburg realization of open string TFT*, hep-th/0305133.

[14] A. Kapustin and Y. Li, *D-branes in Landau-Ginzburg models and algebraic geometry*, *JHEP* **12** (2003) 005 [hep-th/0210296].

[15] A. Kapustin and Y. Li, *Topological correlators in Landau-Ginzburg models with boundaries*, *Adv. Theor. Math. Phys.* **7** (2004) 727 [hep-th/0305136].

[16] D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, math.AG/0302304.

[17] A. Kapustin and Y. Li, *D-branes in topological minimal models: the Landau-Ginzburg approach*, *JHEP* **07** (2004) 045 [hep-th/0306001].

[18] K. Hori and C. Vafa, *Mirror symmetry*, hep-th/0002222.

[19] M.R. Douglas, S. Govindarajan, T. Jayaraman and A. Tomasiello, *D-branes on Calabi-Yau manifolds and superpotentials*, *Commun. Math. Phys.* **248** (2004) 85 [hep-th/0203173].

[20] M. Herbst, C.-I. Lazaroiu and W. Lerche, *Superpotentials, A_∞ relations and WDVV equations for open topological strings*, *JHEP* **02** (2005) 071 [hep-th/0402110].

[21] M. Herbst, C.-I. Lazaroiu and W. Lerche, *D-brane effective action and tachyon condensation in topological minimal models*, *JHEP* **03** (2005) 078 [hep-th/0405138].

[22] A. Recknagel, *Permutation branes*, *JHEP* **04** (2003) 041 [hep-th/0208119].

[23] K. Hori, *Boundary RG flows of $N = 2$ minimal models*, hep-th/0401139.

[24] S.K. Ashok, E. Dell'Aquila and D.-E. Diaconescu, *Fractional branes in Landau-Ginzburg orbifolds*, *Adv. Theor. Math. Phys.* **8** (2004) 461 [hep-th/0401135].

[25] I. Brunner and M.R. Gaberdiel, *Matrix factorisations and permutation branes*, *JHEP* **07** (2005) 012 [hep-th/0503207].

[26] J. Backelin, J. Herzog and H. Sanders, *Matrix factorizations of homogeneous polynomials*, in *Algebra — some current trends*, L.L. Avramov and K.B. Tchakeriam, eds., Lecture Notes in Mathematics **1352**, Springer, Berlin 1988

[27] D.R. Grayson and M.E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, available at http://www.math.uiuc.edu/Macaulay2/.

[28] B. Ezhuthachan, S. Govindarajan and T. Jayaraman, *A quantum McKay correspondence for fractional 2p-branes on LG orbifolds*, *JHEP* **08** (2005) 050 [hep-th/0504164].

[29] I. Brunner and M.R. Gaberdiel, *The matrix factorisations of the D-model*, *J. Phys.* **A 38** (2005) 7901 [hep-th/0506208].

[30] C.I. Lazaroiu, *On the boundary coupling of topological Landau-Ginzburg models*, *JHEP* **05** (2005) 037 [hep-th/0312286].

[31] K. Hori and J. Walcher, *F-term equations near Gepner points*, *JHEP* **01** (2005) 008 [hep-th/0404196].

[32] J. Walcher, *Stability of Landau-Ginzburg branes*, *J. Math. Phys.* **46** (2005) 082305 [hep-th/0412274].

[33] P. Hilton, *Lectures in homological algebra*, in Regional Conference Series in Mathematics **8**, AMS, Providence 1971.

[34] D. Eisenbud, *Homological algebra with an application to group representations*, *Trans. Amer. Math. Soc.* **260** (1980) 35.

[35] L.L. Avramov and D.R. Grayson, *Resolutions and cohomology over complete intersections*, in *Computations in algebraic geometry with Macaulay2* D. Eisenbud, D.R. Grayson, M.G. Stillman and B. Sturmfels eds., Springer, Berlin 2001.

[36] I. Brunner and K. Hori, *Orientifolds and mirror symmetry*, *JHEP* **11** (2004) 005 [hep-th/0303135].

[37] I. Brunner, K. Hori, K. Hosomichi and J. Walcher, *Orientifolds of Gepner models*, hep-th/0401137.

[38] I. Brunner and K. Hori, *Notes on orientifolds of rational conformal field theories*, *JHEP* **07** (2004) 023 [hep-th/0208141].

[39] L. Borisov, M.B. Halpern and C. Schweigert, *Systematic approach to cyclic orbifolds*, *Int. J. Mod. Phys.* **A 13** (1998) 125 [hep-th/9701061].

[40] S.K. Ashok, E. Dell'Aquila, D.-E. Diaconescu and B. Florea, *Obstructed D-branes in Landau-Ginzburg orbifolds*, *Adv. Theor. Math. Phys.* **8** (2004) 427 [hep-th/0404167].

[41] R. Laza, G. Pfister and D. Popescu, *Maximal Cohen-Macaulay modules over the cone of an elliptic curve*, *J. Algebra* **253** (2002) 209.

[42] C.P. Kahn, *Reflexive modules on minimally elliptic singularities*, *Math. Ann.* **285** (1989) 141.

[43] I. Brunner, M. Herbst, W. Lerche and J. Walcher, *Matrix factorizations and mirror symmetry: the cubic curve*, hep-th/0408243.

[44] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge 1993.

[45] P. Mayr, *Phases of supersymmetric D-branes on Kähler manifolds and the McKay correspondence*, *JHEP* **01** (2001) 018 [hep-th/0010223].

[46] A. Recknagel, *On permutation branes*, *Fortschr. Phys.* **51** (2003) 824.